

Mathematical Foundations of Computer Science

Practice Problems for Exam 1

October 7, 2020

1. For sets A, B, C , and D , suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

2. How many sequences of bits are there that have all of the following properties:

- Their length is either 5 or 7 or 9.
- Their middle bit is a 1.
- The number of 0's they have equals the number of 1's they have minus one.

(Give the answer and an explanation of how you obtained it. No proofs required.)

3. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

4. There are a variety of special hands that one can be dealt in poker. For each of the following types of hands, count the number of hands that have that type.

(a) Four of a kind: The hand contains four cards of the same numerical value (e.g., four jacks) and another card.

(b) Three of a kind: The hand contains three cards of the same numerical value and two other cards with two other numerical values.

(c) Flush: The hand contains five cards all of the same suit.

(d) Full house: The hand contains three cards of one value and two cards of another value.

(e) Straight: The five cards have consecutive numerical values, such as 7-8-9-10-jack. Treat ace as being higher than king but not less than 2. The suits are irrelevant.

(f) Straight flush: The hand is both a straight and a flush.

5. Let A, B be arbitrary sets. Prove by contradiction that

$$A \subseteq B \implies A \setminus (A \cap B) = \emptyset.$$

You are NOT allowed to use in the proof *set algebra* facts (such as $A \subseteq B \iff A \cap B = A$ or $A \setminus A = \emptyset$). Your proof should use only the definitions of subset, set difference, intersection, and empty set and logical manipulation of statements.

6. Prove that if for some integer a , $a \geq 3$, then $a^2 > 2a + 1$.

7. Give a combinatorial proof of the following identity for $N, a, b \in \mathbb{N}$:

$$\binom{N}{a} \binom{N}{b} = \sum_{i=0}^{\min(a,b)} \binom{N}{i} \binom{N-i}{a-i} \binom{N-i}{b-i}$$

8. Prove that $\sqrt{6}$ is irrational.

9. There are 100 guests at a fundraising party, excluding the host. As part of a “fun” party game, the host pairs up the dinner guests into 50 pairs that the host calls “fundraising pairs”. In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates \$100 dollars, the individual with the larger net worth must donate \$200 dollars.

The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

10. Prove that for $n \in \mathbb{N}$, with $n \geq 2$, define s_n by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

11. Let n be a positive integer. Prove by induction on n that:

$$\sum_{\{a_1, a_2, \dots, a_k\} \subseteq \{1, 2, \dots, n\}} \frac{1}{a_1 a_2 \cdots a_k} = n$$

(Here the sum is over all non-empty subsets of $\{1, 2, \dots, n\}$. For example, the set $\{1, 3, 6\}$ contributes $\frac{1}{1 \cdot 3 \cdot 6} = \frac{1}{18}$ to the sum.)