Solution Set

CIS 1210—Data Structures and Algorithms—Spring 2025

Stacks, Queues, Heaps—Tuesday, February 18 / Wednesday, February 19

Readings

- Lecture Notes Chapter 13: Stacks & Queues
- Lecture Notes Chapter 14: Binary Heaps and Heapsort

Review: Stacks and Queues

An **abstract data type** (ADT) is an abstraction of a data structure; it specifies the type of data stored and the operations that can be performed, similar to a Java interface. Recall the Stack and Queue ADTs:

Stack	**Queue**
• LIFO (Last-In-First-Out): the most recent element added to the stack will be removed first • Supported operations: – `push`: amortized $O(1)$ – `pop`: amortized $O(1)$ – `peek`: $O(1)$ – `isEmpty`: $O(1)$ – `size`: $O(1)$	• FIFO (First-In-First-Out): the oldest/least recent element added to the queue will be removed first • Supported operations: – `enqueue`: amortized $O(1)$ – `dequeue`: amortized $O(1)$ – `peek`: $O(1)$ – `isEmpty`: $O(1)$ – `size`: $O(1)$

Implementation Details

In this course, we implement stacks and queues using (dynamically resizing) arrays. In other words, we adjust the size of the array so that it is large enough to store all of its current elements but not large enough that it wastes space. The rules we will use for increasing or decreasing the size of a stack or queue's underlying array are as follows:

1. If the array of size n is full, create a new array of size $2n$ and copy all elements into the new array.
2. If the array of size n has less than $\frac{n}{4}$ elements in it, create a new array of size $\frac{n}{2}$ and copy all elements into the new array.

Note that we resize "down" when the array has $\frac{n}{4}$ elements in it (instead of when it has $\frac{n}{2}$ elements) to prevent "thrashing." If we resized "down" when the array has $\frac{n}{2}$ elements, consider the case where we `push` elements onto a stack until it resized "up." If we were to `pop` a single element, then we would have to resize "down," but then if we were to `push` another element, we would have to resize "up" again, so in the worst-case, every `push`/`pop` operation would require copying elements and creating new arrays, increasing our runtimes.

Amortized Analysis

When calculating the runtimes of operations for stacks and queues, we perform amortized analysis. In amortized analysis, the amortized runtime of a single operation is equal to the time needed to perform a series of operations divided by the number of operations performed. For example, let $T(n)$ be the amount of time needed to perform n `push` operations. Then, the amortized runtime of a single `push` operation is equal to $\frac{T(n)}{n}$. Observe that we often perform amortized analysis in situations where the occasional operation takes much longer than the rest of the operations. Considering a stack, in the worst-case, a `push` operation takes $O(n)$ time because of array resizing, but otherwise most of the `push` operations take $O(1)$ time since we're just setting a value at an index of the array.

Note: Amortized analysis is **not** the same as average-case analysis, since it does not depend at all on the probability distribution of inputs. Instead, the total running time of a series of operations is bounded by the total runtime of the amortized operations.

Review: Heaps

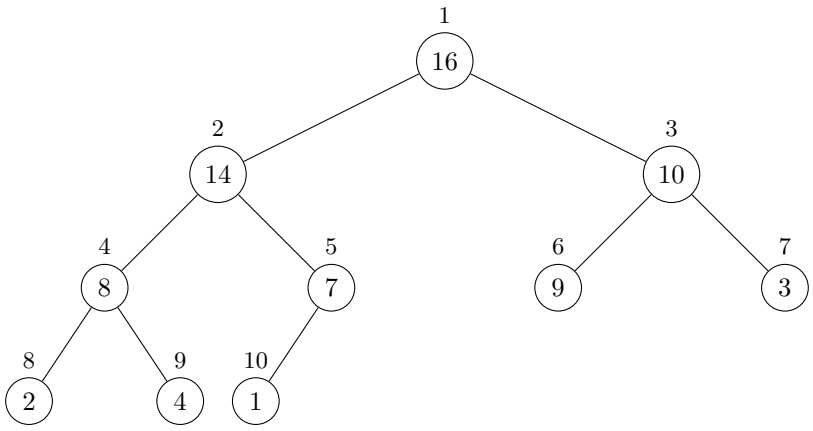
A heap is a tree-like data structure that implements the priority queue abstract data structure (ADT), which allows us to maintain a set of elements, each with an associated key, and select the element with the highest/lowest priority. Heaps satisfy the two following properties:

Heap Property: In a max-heap, for each node i, we have $A[\textsc{Parent}(i)] \geq A[i]$, so the maximum value is stored at the root. In a min-heap, for each node i, we have $A[\textsc{Parent}(i)] \leq A[i]$, so the minimum value is stored at the root.

Shape Property: A heap is an almost complete binary tree, meaning that every level of the tree is completely filled except for the last, which must be filled from left to right.

Implementation Details

Because a heap is an almost complete binary tree, we are able implement it using an array with 1-indexing as shown below:



Index	0	1	2	3	4	5	6	7	8	9	10
Value	`null`	16	14	10	8	7	9	3	2	4	1

Observe that we can populate the array from left to right by doing a level-order traversal of the tree, where we start from the root and go through each level of the tree from left to right. Additionally, because of the shape property, if the root is stored at index 1 of the array, given a node at index i, its left child can be found at index $2i$, its right child can be found at index $2i + 1$, and its parent can be found at index $\lfloor i/2 \rfloor$.

Operations (Max-Heaps)

Max-Heapify maintains the max-heap property at the node called on, so the entire subtree rooted at the node will now be a max-heap. It assumes the node's left and right subtrees are both valid max-heaps and then allows the node to "float-down," swapping it with its larger child or terminating if the max-heap property holds. It runs in $O(h)$ time, where h is the height of the node, since in the worst case, the node must "float down" to the bottom of the tree. Since the height of any node is upper bounded by $\log n$, Max-Heapify runs in $O(\log n)$ time for any node (though this bound may not be tight for some nodes, which is a property we leverage when analyzing the runtime of Build-Max-Heap).

Build-Max-Heap constructs a max-heap from an unsorted array by repeatedly calling Max-Heapify on nodes from the "bottom-up", starting at the nodes right above the leaves (which by definition are max-heaps!). It runs in $O(n)$ time; the mathematical proof of this upper-bound can be found here.

Extract-Max removes and returns the element with the maximum key. We remove the root, replace it with the right-most element in the bottom level/last element in the array, and then call Max-Heapify on the "new" temporary root to maintain the max-heap property. We perform constant work besides calling Max-Heapify, so it runs in $O(\log n)$ time.

Insert adds an element by first adding it to the end of the array/max-heap, and then allowing it to "float-up" to its correct position by repeatedly swapping it with its parent as necessary to maintain the max-heap property. It runs in $O(\log n)$ time, since the path it takes while it "floats-up" has length $O(\log n)$.

Peek returns the maximum element in the heap stored at the root. Since we implement a heap with an array, this runs in $O(1)$ time because we just index into the array.

Problems

Problem 1

You are given two stacks S_1 and S_2 of size n. Implement a queue using S_1, S_2, and a stack's `push`, `pop`, and/or `peek` methods. What are the (amortized) running times of your new `enqueue` and `dequeue` methods?

Solution

`enqueue(x)`:

1. `push x` into S_1.

`dequeue`:

1. If S_2 is empty, `pop` all elements from S_1 and `push` them into S_2. If S_2 is still empty, return Nil.
2. Otherwise, `pop` an element from S_2 and return it.

Proof of Correctness: We want to show that our `enqueue` and `dequeue` methods maintain a queue's FIFO invariant. Since we `enqueue` an element by `push`'ing it onto S_1, to properly `dequeue`, we need to access the elements in S_1 in "reverse" order, from the bottom to the top of the stack. We maintain and ensure this by `pop`'ing elements from S_1 and `push`'ing them onto S_2 when necessary, so we can just `pop` from S_2 to `dequeue`. Since a stack is LIFO, any elements that are pushed into S_2 must be in reverse order relative to how they were pushed into S_1, so popping off S_2 guarantees that the correct element in the queue is retrieved at any time. In the edge case where both S_1 and S_2 are empty, there are no elements in the queue, so we correctly return Nil.

Runtime Analysis: The amortized running time of `enqueue` is $O(1)$, same as a regular stack `push` operation. For the amortized running time of `dequeue`, observe that for each element we `dequeue`, we `push` exactly once (into S_2) and `pop` exactly twice (once from S_1 and once from S_2). Hence, since we have n elements, we

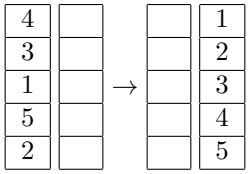
still have an $O(n)$ running time over all `dequeue` operations (as `push` and `pop` are amortized $O(1)$). When we average this over n operations, we see that `dequeue` still runs in $O(1)$ amortized time. At a high-level, when we `dequeue`, note that we only move elements to S_2 if S_2 is empty, and when we move elements onto S_2, we move many elements at once. So, the `dequeue` operation when S_2 is empty pays the "cost," making the following `dequeue` operations faster, since in the future we can `pop` from the now non-empty S_2.

Space Analysis: Beyond the given stacks, we use $O(1)$ additional space to maintain a variable to hold values between the pop and push operation in dequeue.

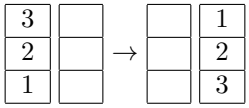
Problem 2

You are given a full stack S_1 with distinct elements and an empty stack S_2, each of size n. Design an algorithm to sort the n elements in increasing order from the top in S_2, using only $O(1)$ additional space beyond S_1 and S_2. What is the running time of your sorting algorithm?

Example:



Hint: Start with a smaller example:



Solution

We use the two given stacks, S_1 and S_2, and two extra variables `max` and `size` in our algorithm.

Algorithm: Initialize `max` to $-\infty$ and `size` to 0. Repeat these steps until `size` $= n$:

1. `pop` all elements from S_1 and `push` them onto S_2. While `pop`'ing, keep track of the maximum element we have seen so far in `max`.
2. `pop` elements from S_2 (until only `size` elements remain in S_2) and `push` all of these elements, except the maximum element stored in `max`, back into S_1.
3. `push` the maximum element (stored in `max`) into S_2.
4. Increment `size` by 1, so we can keep track of the number of sorted elements in S_2 and not `pop` them.

Proof of Correctness: The correctness of our algorithm follows from a stack's LIFO invariant. S_1 starts with all (unsorted) elements, and we maintain this invariant that S_1 only contains elements that have not yet been sorted because in Step 2, we `pop` from S_2 into S_1 so only the bottom `size` elements (the number of elements sorted) remain in S_2. While we `pop` from S_1, we correctly update `max` to be the max element that is currently unsorted and then "sort" this element by `push`'ing `max` into S_2 in Step 3. Our algorithm terminates when `size` $=$ `n` (when S_1 is empty), so all elements have been sorted. Because a stack is LIFO and we "sort" an element each iteration by `push`'ing the maximum unsorted element found into S_2, when our algorithm terminates, S_2 contains all elements sorted in non-decreasing order.

Runtime Analysis: Each iteration of our "loop" (Steps 1 to 4) sorts exactly 1 element. n `push/pop` operations take $O(n)$ time, and for each of the n elements we sort, we `push` and `pop` at most n elements. Therefore, our sorting algorithm runs in $O(n^2)$ time.

Space Analysis: Beyond the given stacks, we use two variables `max` and `size` for $O(1)$ additional space.

Problem 3

Given a data stream of n test scores, design an $O(n \log k)$ time algorithm to find the k-th highest test score. Since PEFS provides minimal monetary resources, CIS 1210 Staff has limited access to storage space and can only afford you $O(k)$ space, where $k \ll n$.

Solution

Algorithm: Construct a min-heap from the first k tests, where tests are ordered by their score, by calling Build-Min-Heap. For each remaining test in the data stream: if its score is greater than the score at the root of the heap, remove the root by calling Extract-Min and then Insert the current test; otherwise, the score of the current test is less than or equal to the score at the root, so do nothing. After processing all tests in the data stream, return the score at the root of the heap by calling Peek.

Proof of Correctness: We will prove the algorithm's correctness by loop invariant. Namely, we will show that we maintain the invariant that the heap always contains the highest k scores we have seen thus far.
Initialization: By calling BuildHeap on the first k, the heap contains the only and thus highest k that have been seen in the stream.
Maintenance: Consider iteration i where we have an element e in the stream. Suppose that the heap contains the highest k until iteration $i-1$. Let m be the result of Peek (the minimum of the heap). There are two cases:

> Case 1: e is in the highest k seen up to iteration i. Thus the minimum of the heap (m) is not in the highest k and can be removed. Because of the loop invariant being held up to this point (highest k until iteration $i-1$), we know that the heap will contain the other $k-1$ highest elements. Thus, the algorithm removing m and inserting e will maintain the loop invariant.
>
> Case 2: e is not in the highest k seen up to iteration i. $e < m$ because e is not in the highest k. The algorithm will not remove element m, which maintains the loop invariant.

Note that in both cases we also maintain the property that the heap holds exactly k scores.
Termination: After the algorithm processes all other $n-k$ elements in the stream, the heap will contain k elements which hold the above invariant.

At the end of the iteration, the heap will contain the k highest scores, the minimum of which will be the k-th highest. This is exactly the score desired.

Runtime Analysis: Constructing a min-heap from the first k tests by calling Build-Min-Heap takes $O(k)$ time. For each remaining test, we either do nothing, or we maintain the heap at size k by calling Extract-Min and then Insert on the current score, which takes $O(\log k)$ time. Each test is inserted into the heap at most once, and since the data stream has n scores, our overall running time is $O(k + n \log k)$. Since $k \ll n$, our final running time is $O(n \log k)$.

Space Analysis: As a "pre-processing" step, we first construct a min-heap with the first k tests. We maintain the heap at size k because for each remaining test that we insert, we remove the root. Therefore, the space complexity of our algorithm is $O(k)$.