- NO OH TODAY

- TAing for PACT (http://algorithmicthinking.org)

    - paid position for 4.5 weeks
    - email me, if interested

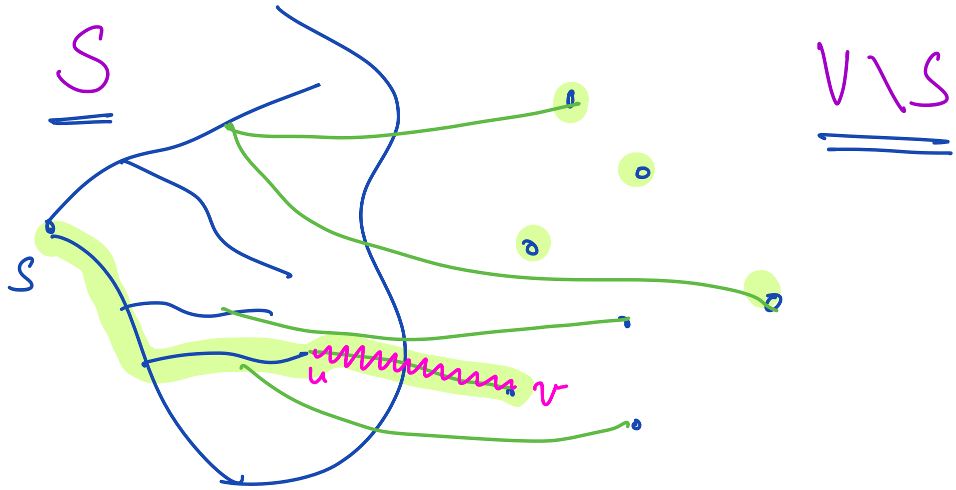
Shortest Paths

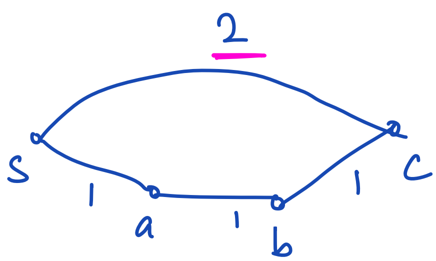
Input: Directed graph $G = (V, E)$

wts on edges : non-negative

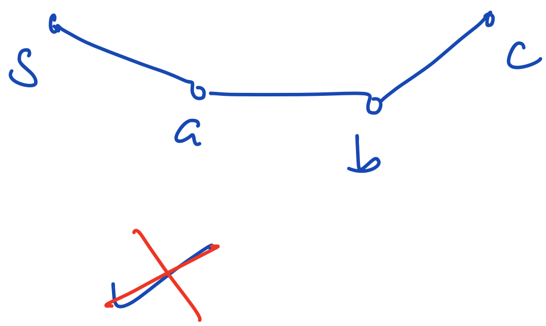
$s \in V$

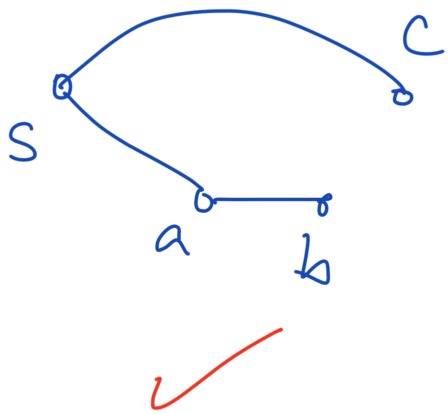
Obj: To find shortest paths from s to every other vertex in G.

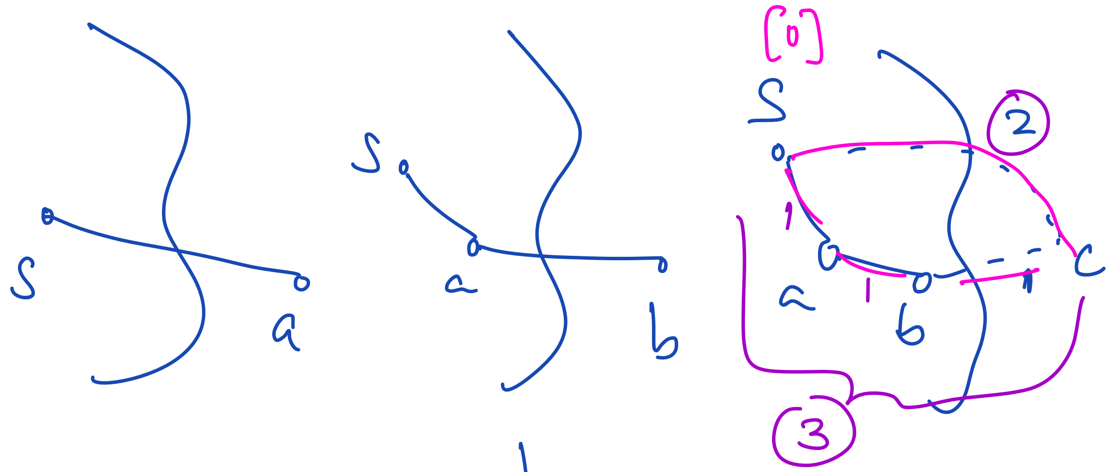


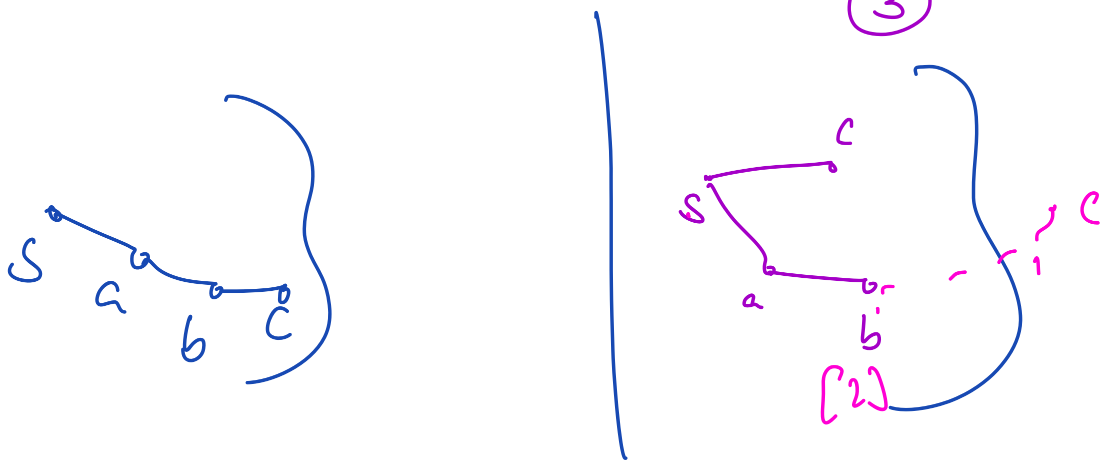


Our o/p









Dijkstra (G, s) // assume WLOG that all vertices in G are reachable from s.

for each $v \in V$ **do**

$d[v] \leftarrow \infty$ // distance estimate

$\pi[v] \leftarrow$ NIL

Heap

keys : $d[\cdot]$

BuildHeap : $O(n)$

$*$ $\boxed{d[s] \leftarrow 0}$

$S \leftarrow \emptyset$

while $S \neq V$ **do** ← n iterations.

$O(n)$ $u \leftarrow$ vertex in $V \setminus S$ with the smallest $d[\cdot]$. $\circledast$ → ExtractMin $O(\lg n)$

$S \leftarrow S \cup \{u\}$

$O(m)$ for each $v \in N(u) \cap \{V \setminus S\}$:

if $d[v] > d[u] + w_{uv}$ then

$d[v] \leftarrow d[u] + w_{uv}$ ← Decreasekey $O(\lg n)$

$\pi[v] \leftarrow u$

Running time : $\cancel{O(n+m)}$

$O(n^2+m)$ ←

$O((n+m)\lg n)$

$= O(m \lg n)$ ←

Correctness

Theorem : Dijkstra's alg. gives us shortest paths correctly.

Proof : Induction on $|S|$.

IH : Assume that the claim holds when $|S| = k$. That is, assume that our alg. gives shortest paths correctly for each vertex in S when $|S| = k$.

BC : $|S| = 1$

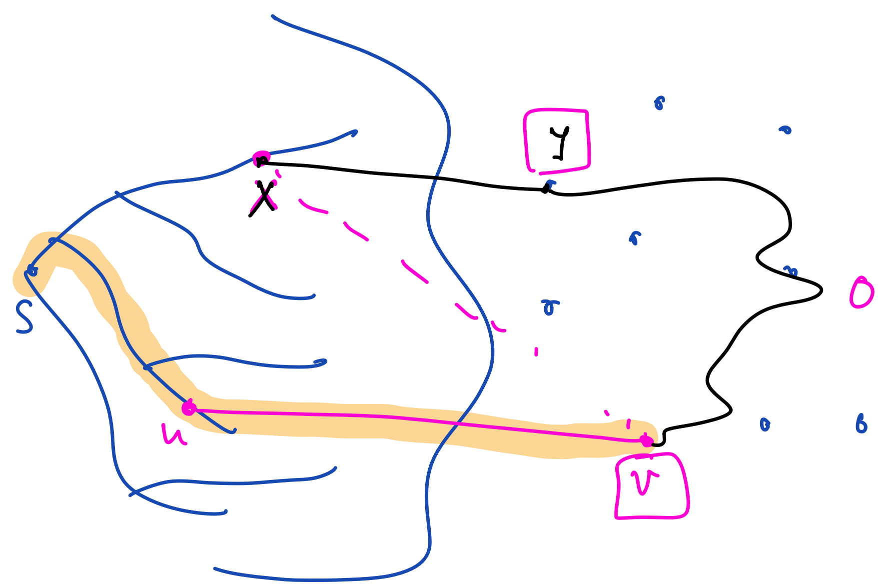
$S \ni \{s\}$

$d[s] = 0$ ✓

IS: We want to prove that the claim holds when $|S| = k+1$. Let v be the $(k+1)^{th}$ vertex brought into S. Let $\pi[v] = u$. That is, $d[v] = d[u] + w_{uv}$.

Assume for contradiction that $s \rightsquigarrow u \rightarrow v$ is not the shortest $s \rightsquigarrow v$ path. Instead, the shortest path is

$$s \rightsquigarrow x \rightarrow y \rightsquigarrow v.$$



Since length of the path

$s \rightsquigarrow x \rightarrow y \rightsquigarrow v$ is shorter than

true only if edge wts are non-neg.

$d[v]$, $d[y] < d[v]$, &

hence y would be the

$(k+1)^{th}$ vertex brought into S

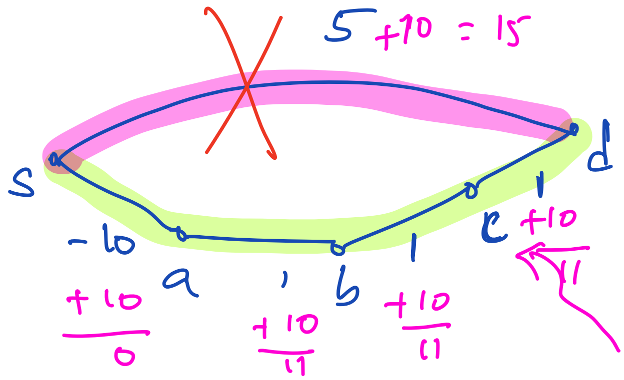
& not v, a contradiction!

Natural soln when edge wts are -ve.

Add the most negative edge wt to

all edges. Now all edge wts are

non-neg. Run Dijkstra.



Shortest path from s to d.

Minimum Spanning Trees

Input: Undirected graph $G = (V, E)$

wts on edges : positive

Obj : To find a min wt spanning subgraph of G, that is connected.

↳ all vertices & no edges.

Lemma: The output must be a tree.

Assumption: all edge wts are distinct.

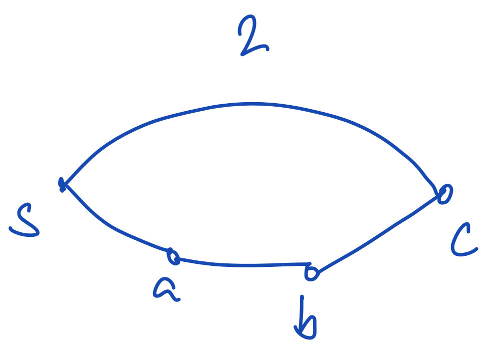
Alg:

- Reverse Delete
  - Process edges in $\downarrow$ order of their wts.
  - Remove the edge if removing it does not disconnect the graph.
- Kruskal's

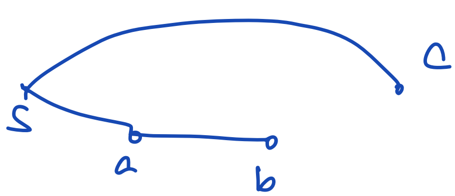
- Process edges in ↗ order of wts

- Add the edge if adding it does not create a cycle.

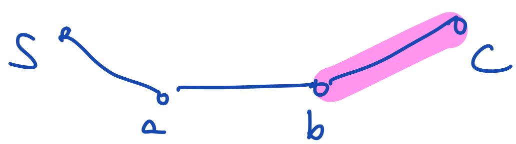
- ~~Dijkstra~~ "wrong Dijkstra" aka Prim's alg.



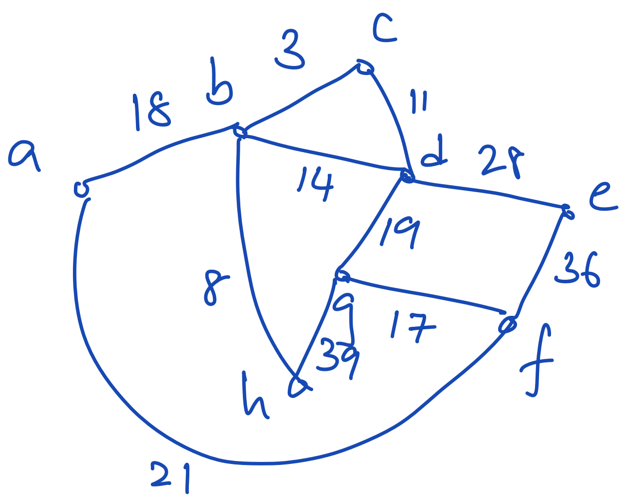
Dijkstra

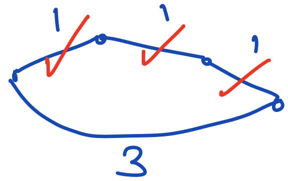


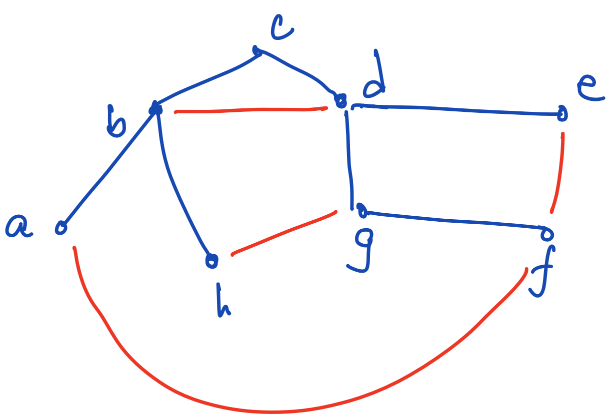
MST



Examph







Prim

$d[v] > w_{uv}$

if $\cancel{d[v] > d[u] + w_{uv}}$ then

$d[v] \leftarrow w_{uv}$

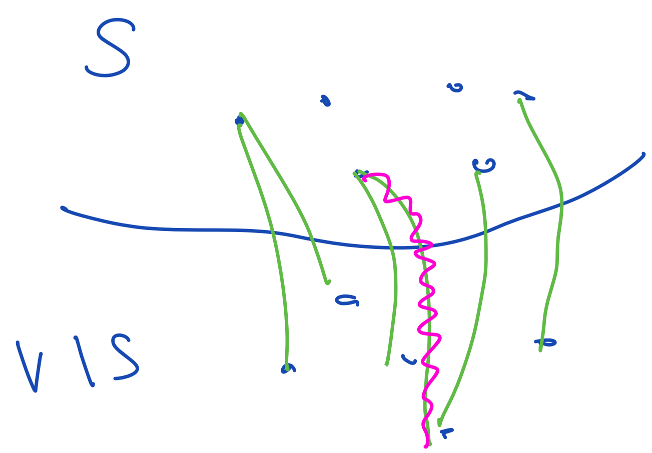
$\cancel{d[v] \leftarrow d[u] + w_{uv}}$

$\pi[v] \leftarrow u$

Lemma: Let $(S, V \setminus S)$ be a partition of vertices in V s.t. $S \neq \phi$ & $S \subset V$.

Let $e = (u, v)$ be an edge that crosses $(S, V \setminus S)$, s.t. among all edges that cross $(S, V \setminus S)$, e has the smallest wt.

Then e must be in every MST.



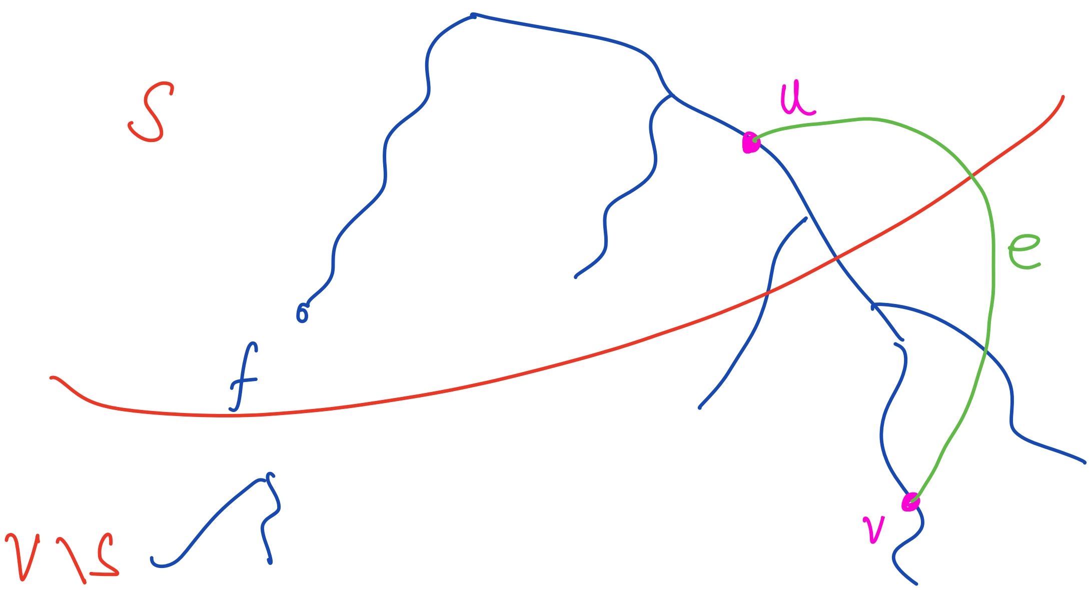
Proof: Assume for contradiction that there is a MST T that does not contain e. Let f be an edge in T that crosses $(S, V \setminus S)$.

Since e is the smallest wt edge crossing $(S, V \setminus S)$, $w_e < w_f$.

Consider

$$T' = T \setminus \{f\} \cup \{e\}$$

Clearly, T' is a spanning tree of a wt lower than wt(T), contradicting that T is a min wt. spanning tree.



Choose f to be the edge on the $u \sim v$ path in T that crosses the cut $(S, V \setminus S)$. We know f must exist.