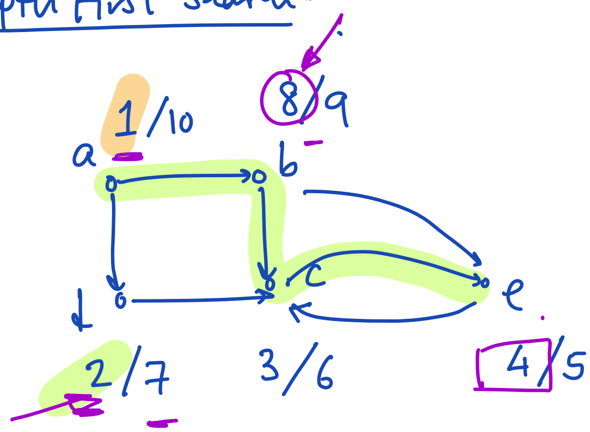
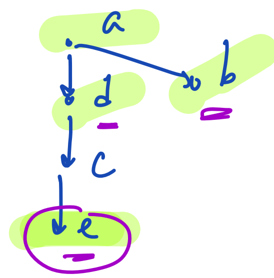
- NO OH TODAY

- Exam 1
  - Seating on class page
  - reiterate lectures / hws / recitations

Depth First Search.




d[u] : time at which vertex u is discovered.

f(u) : " " " " " & its neighbors are done exploring.

color(u) : white $\longrightarrow$ gray $\rightarrow$ black

Running time: $O(n+m)$

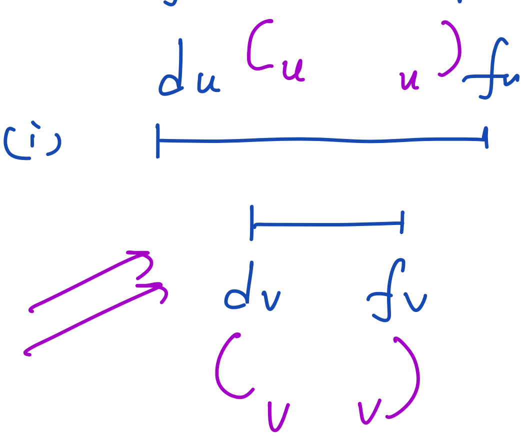
## Properties of DFS

When is a vertex $v$ a descendant of vertex $u$ in the DFS forest?

**Property 1** : $v$ is a descendant of $u$ in the DFS forest iff $v$ is discovered when $u$ is gray.

**Property 2 (Parenthesis theorem)** : Let $u$ & $v$ be any two vertices in $G$. Then exactly one of the following happens:

(i)

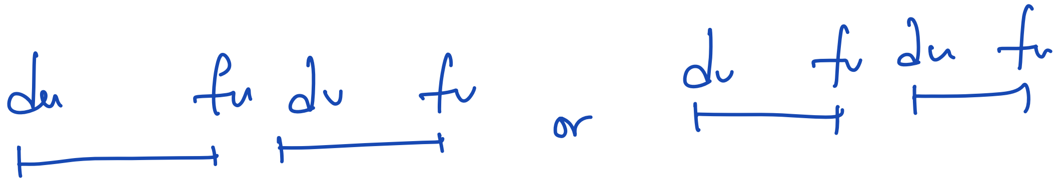



& $v$ is a descendant of $u$ in the DFS forest.

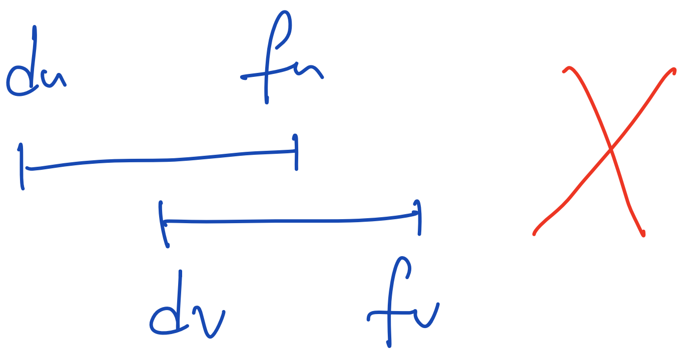
$(_u \ (_v \ _v) \ _u)$

(ii) 


& u is a descendant of v in the DFS forest.

(iii) 


& neither vertex is a descendant of the other ~~in~~ the DFS forest.

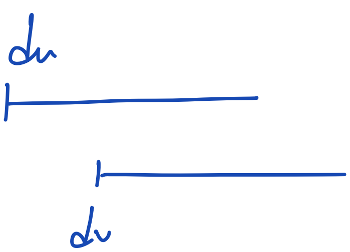



<u>Proof sketch</u> : WLOG, let $d_u < d_v$.

<u>Case I</u> : $d_v > f_u$


Since v is discovered when u is black, v is not a descendant of u in the DFS forest. Similar reasoning to argue that u is not a descendant of v in the DFS forest.

<u>Case II</u> : $d_v < f_u$




Since v is discovered when u is gray, by property 1, v is a descendant of u in the DFS forest.

It remains to show that $f_v < f_u$.

when DFS is at vertex v, it finishes exploring all neighbours of v, colors v black before the search goes back to u. Hence $f[v] < f[u]$.

<u>Corollary</u>: Vertex v is a descendant of vertex u in the DFS forest iff

(*)

$$d_u < d_v < f_v < f_u.$$

<u>Property 3 (White Path Theorem)</u>

Vertex v is a descendant of u in the

DFS forest iff at d[u] there is a

white path (path consisting of white vertices) from

u to v in G.

<u>Proof</u> : ($\Rightarrow$) v is a descendant of u in

the DFS forest $\Rightarrow$ at d[u] there is a

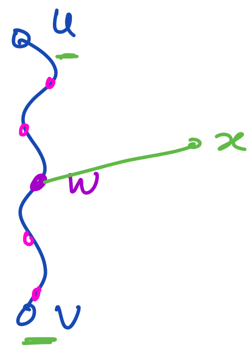
white path from u to v in G.

Let P be the path from u to v in the

DFS forest.

w : any vertex in P.

P :




By property 1 (even property 2 will work),

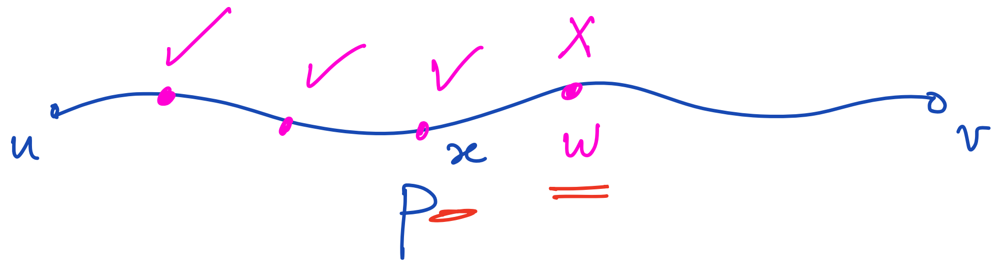
w is discovered when color[u] is grey,

which means that at time $d[u]$, color$[w]$ is white.

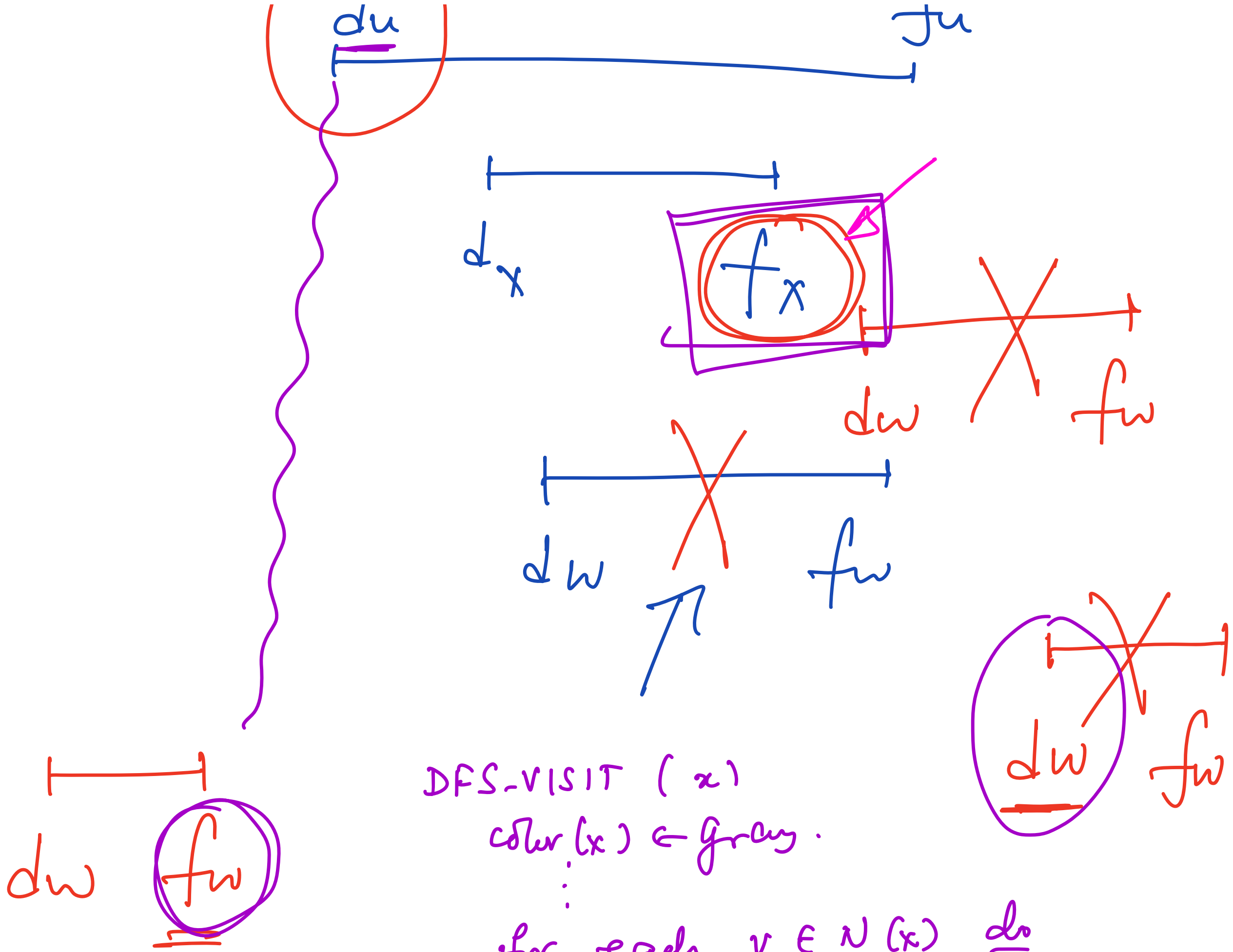
($\Leftarrow$) at time $d[u]$ there is a white path from $u$ to $v$ in $G$ $\Rightarrow$ $v$ is a descendant of $u$ in the DFS forest.

<u>Proof</u>: Assume for contradiction that at time $d[u]$ there is a white path from $u$ to $v$ in $G$, but $v$ is not a descendant of $u$ in the DFS forest. Let $P$ be the white

path from $u$ to $v$ in $G$ at time $d(u)$.




Going from $u$ towards $v$ along $P$, let $w$ be the first vertex that is not a descendant of $u$ in the DFS forest. Let $x$ be the vertex just before $w$ in $P$. $x$ is a descendant of $u$ in the DFS forest.




```
DFS-VISIT (x)
   color(x) ← gray.
      ⋮
   for each v ∈ N(x) do
      if color(v) = white then
         π[v] ← x
         DFS-VIST(v)
   color(x) ← Black.
```

Note that the interval [dw, fw] cannot be contained in [du, fu] (by the Parenthesis thm)

$[d_w, f_w]$ cannot be "after" $f_u$,

i.e., $d_w > f_u$ is not possible because

$w$ is a neighbor of $x$ &

$x$ cannot have a white

neighbor when it finishes.

Thus $f_w < d_u$, contradicting that there

is a white path from $u$ to $w$ at time

$d[u]$.

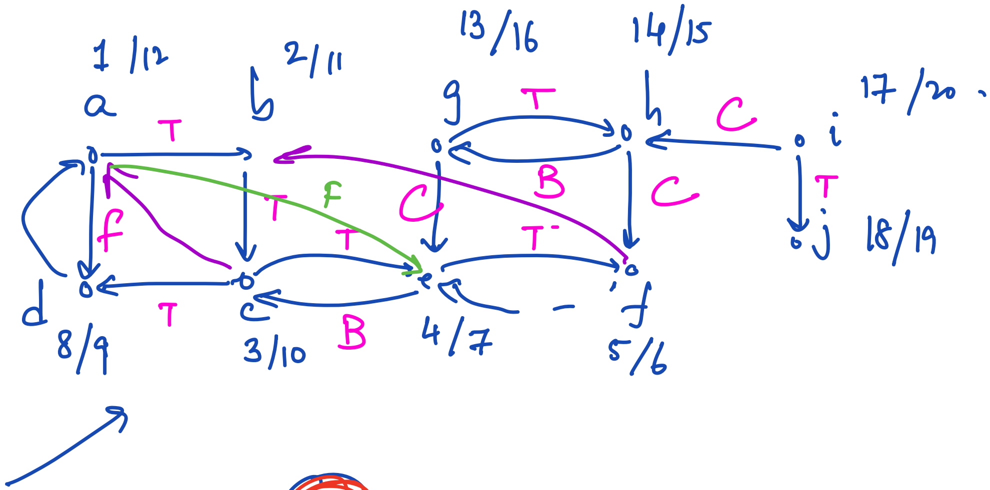
**Edge Classification**: During DFS, we can label each edge of G as follows. Note that each edge is labeled the very first time it is explored. Edge $e = (u, v)$ is a:

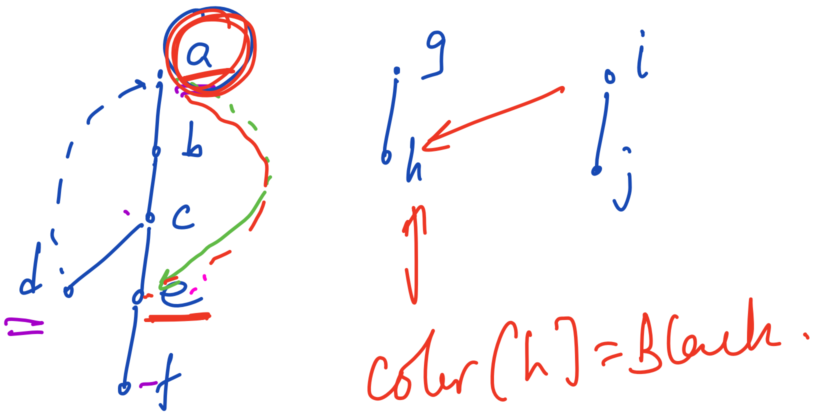
**tree edge**: if e is in the DFS forest.

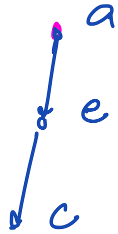
**back edge**: if v is an ancestor of u in the DFS forest.

**forward edge**: if v is a descendant of u in the DFS forest.

**Cross edge**: if it is none of the above.




Suppose $e = (u,v)$ is a tree edge. Then when e is explored first, color[v] = white.

e = (u,v) is a back edge. Then when e is explored first color[v] = Gray.

e is a forward edge. Then when e is explored first, color[v] = Black.

Theorem: e = (u,v) is a forward edge ~~iff~~ when e is explored first, color[v] = Black & v is a descendant of u in the DFS forest.

Theorem: DFS on an undirected graph G yields Tree edges

Back edges

Forward edges

Cross edges.

Proof: Let $e = (u,v)$ be any edge in $G$.

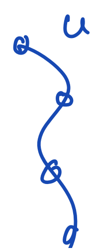
WLOG, let $d_u < d_v$.

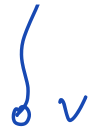
Claim: $v$ is a descendant of $u$ in the DFS forest.

Case I: $v$ is a child of $u$ in the DFS forest.

$e$ is a tree edge.

Case II: $v$ is a descendant of $u$ in the DFS forest, but $v$ is not a child of $u$.






e is first explored when the search is at v. At that time u is an ancestor of v in the DFS forest & hence e is a <u>back edge</u>.