CIS 1210 — Data Structures and Algorithms
Homework Assignment 1

Assigned: January 16, 2025 **Due:** January 27, 2025

Note: The homework is due **electronically on Gradescope** on January 27, 2025 by 11:59 pm ET. For late submissions, please refer to the Late Submission Policy on the course webpage. You may submit this assignment up to 2 days late.

A. **Gradescope**: You must select the appropriate pages on Gradescope. Gradescope makes this easy for you: before you submit, it asks you to associate pages with the homework questions. Forgetting to do so will incur a 5% penalty, which cannot be argued against after the fact.

B. **LaTeX**: You must use the LaTeX template provided on the course website, or a 5% penalty will be incurred. Handwritten solutions or solutions not typeset in LaTeX will not be accepted.

C. **Solutions**: Please write concise and clear solutions; you will get only a partial credit for correct solutions that are either unnecessarily long or not clear. Please refer to the Written Homework Guidelines for all the requirements. Ed Discussion will also contain a complete sample solution in a pinned post. If applicable, all your answers should be in closed form, unless otherwise specified.

D. **Algorithms**: Whenever you present an algorithm, your answer must include 3 separate sections. Please see Ed Discussion for an example complete solution.

1. A precise description of your algorithm in English. No pseudocode, no code.
2. Proof of correctness of your algorithm
3. Analysis of the running time complexity of your algorithm

E. **Collaboration**: You are allowed to discuss **ideas** for solving homework problems in groups of up to 3 people but *you must write your solutions independently.* Also, you must write on your homework the names of the people with whom you discussed. For more on the collaboration policy, please see the course webpage.

F. **Outside Resources**: Finally, you are not allowed to use *any* material outside of the class notes and the textbook. Any violation of this policy may seriously affect your grade in the class. If you're unsure if something violates our policy, please ask.

Collaboration policy for written homeworks: You are allowed to **discuss** ideas for solving homework problems in groups of three, documenting who you discussed with at the top of your assignment. You are **not** allowed to write up the solutions together.

- It is fine to discuss the topics covered in the homeworks, to discuss approaches to problems, and to sketch out general solutions. However, you MUST write up the homework answers and solutions individually. You are not permitted to share specific solutions, mathematical results, etc.
- Our suggestion is to discuss the problems together, but if you made any notes or worked out something on a white board with another person while you were discussing the homework, then erase or destroy those notes as soon as the discussion is over. You shouldn't use those notes while writing up your answer, however tempted you may be to do so. This will force you to write up the solutions yourself and to make sure that you genuinely and fully understand them.

Any violation of the collaboration policy will be dealt with severely.

*** As you saw in the lecture notes, the stable-matching problem was originally formulated as a marriage problem between n men and n women. Obviously this 1950's view of dating/marriage is quite naive and narrow-minded. Nevertheless, this formulation of the problem provides clear notation and is relatively simple and unambiguous to talk about. For this reason, the lecture notes and some of the problems in this homework use this formulation of the stable-matching setup. Please know we don't intend for this to be insensitive or rude, it is just the standard way to present the problem. If you would like, you may reword the stable matching setup, though you must be clear which set of objects is the proposing side. ***

1. [20 pts] Back to the Slopes of CIS 1600

At the ski resort hosting the annual CIS 1210 winter retreat, each TA is assigned their own ski lodge. Since the weather can make skiing difficult, some ski lodges are connected by bidirectional ski lifts. As the leader of the ski retreat, Patrick needs to answer the following questions about the network of ski lodges below.

You may find the following definitions helpful:

- $\delta(G)$ refers to the minimum degree of any vertex in the graph G
- $\chi(G)$ refers to the chromatic number of the graph, i.e. the minimum number of colors needed to color the graph such that no two adjacent vertices have the same color
- An induced subgraph of G is a graph formed from a non-empty subset of the vertices of G and all of the edges in G connecting pairs of vertices in that subset.

a. For this part only, assume that each ski lodge is directly connected to at most k other ski lodges. Prove by construction that it is possible to use $k+1$ types of flags to mark the ski lodges such that no two adjacent ski lodges have the same type of flag.

b. For this part only, Patrick notices that the average number of connections that each ski lodge has is at most k. Prove or disprove that it is always possible to mark the ski lodges with $k+1$ types of flags such that no two adjacent ski lodges have the same type of flag.

c. Patrick observes that the network of ski lodges can be represented as a graph G, where the ski lodges are vertices and the ski lifts are edges. Patrick also notices that $\delta(H) \leq d$ for all induced subgraphs H of G and for some value $d \in \mathbb{N}$. Prove or disprove that $\chi(G) \leq d + 1$.

2. [20 pts] An Opportune Observation

The snow has been particularly heavy this year. Darsh has decided that instead of rotting away on Instagram Reels, he would hit the slopes. While going up the lifts, he noticed many kids playing hide and seek on the mountain. Let $H = \{h_1, h_2, \ldots, h_{n-1}, h\}$ be the set of n unique hiders and $S = \{s_1, s_2, \ldots, s_{n-1}, s\}$ be the set of n unique seekers.

- s is an incredibly smart seeker who was ranked last by every hider in H (i.e. the hiders think they are the most difficult to run away from).
- There is one particular hider $h \in H$ that was ranked second by every seeker in S.
- Each seeker except for s (that is, $s_i,\ 1 \leq i \leq n-1$), has a unique first preference that is not h.

Darsh, equally a CIS nerd as a hide and seek enthusiast, began to question what stable matching would be obtained by the Gale-Shapley algorithm under two scenarios: when S is the proposing side and when H is the proposing side. Help Darsh out by determining the stable matching(s). Prove your answer.

3. [20 pts] Sorting Ski Lifts

On a CIS 1210 ski trip, the TAs have to decide which ski lift to take up the mountain. As the trusty head TA, Gabe is forced to split the TAs into groups. Because he's been to this resort before, Gabe creates a ranking for each of the n ski lifts in order of how much he thinks the m TAs will enjoy it. To keep track of everyone, all the TAs grouped on a ski lift must fit on the same ski lift chair where the k^{th} ski lift's chair can hold up to n_k TAs. Additionally, the TAs have done their own research and have created rankings of the n ski lifts, which they give to Gabe.

Note that each TA can only go into at most one group, and that it is not necessarily true that $\sum n_k = m$. Since Gabe is determined to please the staff, he will only be satisfied if he can group the TAs such that there is no situation where a TA a is sorted onto ski lift b if TA a prefers another ski lift b' and Gabe prefers to give ski lift b' to TA a over one of the TAs currently in the group.

Describe how Gabe could use the Gale-Shapely algorithm (you should not modify the algorithm) to distribute the TAs in a way that satisfies him. Justify the correctness of your solution.

4. [20 pts] The Ultimate CIS 1210 Underground Snowball Fight

One of Erica's new year's resolutions is masterminding the most intense challenge of 2025: the CIS 1210 underground snowball fight. Historically, her resolutions have never quite worked out for her, because she didn't have your support! Answer the following questions so Erica can finally have a great start to her year.

a. The underground snowball fight battleground consists of n snowball stations arranged in a circle, and at most one bidirectional tunnel connecting two distinct stations. Each of the n TAs is assigned a unique snowball station. Moreover, Erica decides to add a tunnel between any two snowball stations with probability of $\frac{1}{2}$. She begins exploring the battleground, such that she starts from some TA's assigned snowball station, runs through a unique set of tunnels (without visiting the same snowball station twice), and gets back to the snowball station she started from. Soon, she begins to wonder what the expected number of such cycles there are in her battleground, such that the number of unique tunnels she runs through to get back to the original snowball station is exactly k. Help Erica solve this, or the CIS 1210 snowball fight will never happen! You can assume that $2 \leq k \leq n$.

b. Erica finally finished setting up for the underground snowball fight, and now the TAs are ready to begin fighting. Kimberly is feeling extra devious today though, and decides that instead of running through the tunnels that Erica meticulously designed earlier, she will go above ground and steal one unlucky TA's snowballs, stopping after she steals. Since the snowball stations are arranged in a circle, Kimberly runs clockwise, passing through each station one by one. Every time she gets to a snowball station, she decides with independent probability p to steal that TA's snowballs. Note that if Kimberly completes the circle, she keeps running (thus, she may pass through any snowball station multiple times). If Erica is k^{th} in the circle (assume $1 \leq k \leq n$), what is the probability that her snowballs will be stolen and her second New Year's resolution of winning the fight will be crushed?

5. [20 pts] Skiing School

Eloic runs the most expensive skiing academy in the world as his side hustle. Today, he has n students show up and wants to pair each one up with one of the n instructors for a private lesson. To do so, each student comes up with a preference order of which instructor they want to learn from, and each instructor comes up with a preference order of which student they want to teach. Eloic, having access to all the preference lists, then uses the Gale-Shapley algorithm to pair up the students and instructors. Specifically, when going down the slopes, each student will go down their preference list and find the instructor that they like the most until one decides to teach them. The instructor will continue down the slopes with them until they are asked to teach another student they prefer more, in which they will begin instructing the preferred student instead.

Eloic knows that Aaron Mei, a student at his very first ski lesson, is really nervous about going down the slopes. To ease his nerves, Eloic wants to make it so that the first instructor Aaron asks to teach him will stay with him for the entire slope. Help Eloic design an algorithm that proposes a modification to Aaron's original preference list, such that when the students start pairing up with an isntructor, the first instructor Aaron asks never leaves him. Remember to **explicitly state your algorithm**, **analyze its runtime**, and **prove its correctness**.

Hint: It may be helpful to prove that the GS algorithm outputs a stable matching when there are fewer students than instructors.