CIS 1210 — Data Structures and Algorithms
Homework Assignment 1

Assigned: August 27, 2024 **Due:** September 9, 2024

Note: The homework is due **electronically on Gradescope** on September 9, 2024 by 11:59 pm ET. For late submissions, please refer to the Late Submission Policy on the course webpage. You may submit this assignment up to 2 days late.

A. Gradescope: You must select the appropriate pages on Gradescope. Gradescope makes this easy for you: before you submit, it asks you to associate pages with the homework questions. Forgetting to do so will incur a 5% penalty, which cannot be argued against after the fact.

B. LaTeX: You must use the LaTex template provided on the course website, or a 5% penalty will be incurred. Handwritten solutions or solutions not typeset in LaTex will not be accepted.

C. Solutions: Please write concise and clear solutions; you will get only a partial credit for correct solutions that are either unnecessarily long or not clear. Please refer to the Written Homework Guidelines for all the requirements.

D. Algorithms: Whenever you present an algorithm, your answer must include 3 separate sections.

1. A precise description of your algorithm in English. No pseudocode, no code.
2. Proof of correctness of your algorithm
3. Analysis of the running time complexity of your algorithm

E. Collaboration: You are allowed to discuss **ideas** for solving homework problems in groups of up to 3 people but *you must write your solutions independently*. Also, you must write on your homework the names of the people with whom you discussed. For more on the collaboration policy, please see the course webpage.

F. Outside Resources: Finally, you are not allowed to use *any* material outside of the class notes and the textbook. Any violation of this policy may seriously affect your grade in the class. If you're unsure if something violates our policy, please ask. If you would like to cite from CLRS in your proofs you must first ask for permission on ED.

1. [10 pts] Networking Nuisance

Gabe is building a secure network to ensure that all Olympic streaming can continue running despite the global cybersecurity shutdown. He has n servers ($n \geq 2$), which are interconnected by bidirectional ethernet cables. At most one cable connects any two servers. In order to ensure that a malicious attack is contained, he consciously chooses to connect the servers such that **not** every server can be reached from every other server. Prove that there exists a pair of servers such that the total number of ethernet cables leading out of the pair is less than $n - 1$.

2. [30 pts] Max's Flag Formations

You may find the following definitions helpful:

- $\delta(G)$ refers to the minimum degree of any vertex in the graph G
- $\chi(G)$ refers to the chromatic number of the graph, i.e. the minimum number of colors needed to color the graph such that no two adjacent vertices have the same color
- An induced subgraph of G is a graph formed from a non-empty subset of the vertices of G and all of the edges in G connecting pairs of vertices in that subset.

a. While training for Olympic skateboarding this summer, Max decides to set up a series of n flags around his training course to practice skating around. Each of the n flags is connected to at most k other flags with a bidirectional path (that is, flag u is connected to flag v if and only if v is connected to u). He wants to use different colors for adjacent flags, but isn't sure how many colors he needs. Prove by construction that it is possible for the n flags to each be colored one of $k+1$ colors such that no two connected (*adjacent*) flags have the same color.

b. Max noticed that the average flag in his training course is *adjacent* to at most k other flags - if we sum over the number of adjacent flags for each flag and divide it by n we get a value less than or equal to k. Prove or disprove that it is always possible to use a flag with one of the $k+1$ colors such that no two adjacent flags have the same color.

c. Impressed with his training course, the USA sailing team commissioned Max to design a training course for them (so they can all win Olympic gold). Max similarly constructed a network of n buoys connected by bidirectional paths, which can be expressed as a graph G, where every buoy is a vertex and adjacent buoys are connected with edges. Max constructs the training courses such that $\delta(H) \leq d$, for all induced subgraphs H of G for some $d \in \mathbb{N}$. Prove or disprove that the n buoys can be painted in $d+1$ colors such that no two adjacent buoys are the same color (this is to say that $\chi(G) \leq d+1$).

3. [20 pts] Anand's Spectacular Routine

Anand is preparing for his Olympic Parallel Bars Gymnastics Routine and is trying to determine which techniques to include. He has mastered a total of 40 skills which he initially order from least to most difficult in a sequence from 1 to 40. He then decides to shuffle the order of the skills by picking a random location in this sequence to "flip" the techniques. Note that a "flip" consists of picking a number x uniformly at random (where $1 \leq x \leq 40$), and taking all skills numbered less than x and moving them to the end of the routine. For instance the following demonstrates a flip at skill number 3 (note that the start of the routine is on the left):

$$1, 2, 3, 4 \rightarrow 3, 4, 1, 2$$

Now, Anand wants to know how many of his difficult skills are going to stay at the end of the routine. Let the difficult skills of the routine refer to the last 20 skills in the routine, and let X represent the number of skills that were in the second half of the routine before any flip(s) occurred, that remained in the second half of the routine after the flip(s) have occurred.

(a) What is $\mathbf{E}[X]$ when there is one flip? Justify your answer.

(b) Now suppose Anand flips the routine n times in random positions, where $1 \leq n \leq 40$. In that case, what is $\mathbf{E}[X]$? Justify your answer.

4. [20 pts] Kimberly's Olympic Ordering

Kimberly, as a big fan of the Olympics, wants to curate a list of every single Olympian that competed in the games this summer. She wants to organize the list of names in alphabetic order and, since she is also a big computer science nerd, decides to store them all in a binary search tree (BST) data structure of type OlympianNode. However, she was too distracted by watching the replays that she accidentally arranged them in *reverse alphabetical order*! Given the following interface for the nullable class OlympianNode, help Kimberly fix her mistake and organize the names in alphabetic order!

```
OlympianNode():
    Name : string // the name of the Olympian stored in this node
    Left : OlympianNode // pointer to the left subtree in the BST
    Right : OlympianNode // pointer to the right subtree in the BST
```

(a) Help Kimberly design a $\Theta(n)$ algorithm to print out all names stored in this BST in alphabetical order. You may assume that the given root node is non-null. **Hint:** you may find recursion useful here.

(b) Can she create an asymptotically more efficient algorithm to accomplish her task? Explain why or why not.

5. [20 pts] Technical Tennis Troubles

After training for her entire life, Erica is approaching the pinnacle of her Tennis career. She is about to compete in the Women's Singles Tennis finals at the Olympics and has a shot at gold. But disaster... while sitting in the locker room, she has started experiencing pre-competition nerves because she cannot figure out the following problems.

(a) If x and y are both divisible by k, prove that $\gcd(x, y) = k \cdot \gcd(\frac{x}{k}, \frac{y}{k})$, where $k > 1$.

(b) Let $gcd(a, b)$ be the input to Euclid's Algorithm, shown in class, where $a > b$. Give the worst-case recurrence relation for this algorithm as a recurrence function $T(n)$, where $n = min(a, b)$. Don't forget to state the base case.

By solving these questions, you can help Erica alleviate her stress so she can focus on her match and win gold, bringing glory to CIS 1210!