Convergence of No-Regret Play to Nash Equilibrium

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- But you need to understand the game extremely well and make careful calculations.
- Is there a natural dynamic that leads to Nash equilibrium if everyone uses it?
- How many of these properties depend on the "two player" caveat?

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The answer is no.

"Meta Theorem": n player zero-sum games don't have any special properties that $n-1$ player general sum games don't have.

In particular, we should not expect such games to have a value, nor that their equilibria should be easy to compute.

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"Proof": Any $n-1$ player game can be made into an n player zero sum game, by adding a new player n (with a trivial action set), and $u_n(a) = -\sum_{i=1}^{n-1} u_i(a)$. Since player n is payoff irrelevant to the $n-1$ other players, the equilibrium structure remains identical to the original game.

But we can generalize with more structure...

Definition

A separable graphical game is defined by a graph $G = (V, E)$. The set of players corresponds to the set of vertices: $P = V$. Each player's utility function is decomposable as a sum of neighbor-specific utility functions, one for each of his neighbors in G:

$$u_i(a) = \sum_{(i,j) \in E} u_i^{(i,j)}(a_i, a_j)$$

i.e. it is as if each player is playing a 2-player game with each of his neighbors – except he must pick a single action a_i to play simultaneously against each of his neighbors.

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1. They continue to have a *value*
2. Equilibria are easy to compute with efficient dynamics.
3. We don't require each of the constituent 2-player games are zero sum — just that the aggregate is.

Regret

Definition

A sequence of action profiles $a^1, \ldots, a^T$ has regret $\Delta(T)$ if for all players i and actions a_i^* we have:

$$\frac{1}{T}\sum_{t=1}^{T} u_i(a^t) \geq \frac{1}{T}\sum_{t=1}^{T} u_i(a_i^*, a_{-i}^t) - \Delta(T)$$

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3. But not the only way...
4. A permissive *family* of dynamics.

Dynamics

Given a sequence of action profiles $a^1, \ldots, a^T$, write $\bar{a}_i = \frac{1}{T}\sum_{i=1}^{T} a_i^t$ to denote the mixed strategy for player i that selects an action in $\{a_i^1, \ldots, a_i^T\}$ uniformly at random.

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Theorem

Consider any zero sum separable graphical game G. If a sequence of action profiles $a^1, \ldots, a^T$ has regret $\Delta(T)$, then the mixed strategies:

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If every player plays using polynomial weights, they converge to an ϵ-approximate Nash equilibrium by in:

$$T = \frac{4n^2 \log k}{\epsilon^2}$$

many rounds. In a two player game this is $T = 16\log(k)/\epsilon^2$ steps.

Proof

1. A useful fact: for every action $a_i^* \in A_i$ we have:

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\sum_{(i,j)\in E} u_i^{i,j}(a_i^*, a_j^t) &= \sum_{(i,j)\in E}\sum_{t=1}^{T}\frac{1}{T}u_i^{i,j}(a_i^*, a_j^t)\\ &= \sum_{(i,j)\in E} u_i^{i,j}(a_i^*, \bar{a}_j)\end{aligned}$$

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$$= \sum_{(i,j)\in E} u_i^{i,j}(a_i^*, \bar{a}_j)$$

2. Suppose every player i is playing according to $\bar{a}_i$. Let a_i^* be the best response of player i to the distribution of his opponents. We know:

$$\sum_{(i,j)\in E} u_i^{i,j}(a_i^*, \bar{a}_j) \geq \sum_{(i,j)\in E} u_i^{i,j}(\bar{a}_i, \bar{a}_j)$$

Proof

1. We also know, since $a^1, \ldots, a^t$ have $\Delta(T)$ regret, that for all $i \in P$:

$$\underbrace{\frac{1}{T}\sum_{t=1}^{T}\sum_{(i,j)\in E} u_i^{(i,j)}(a_i^t, a_j^t)}_{LHS} \geq \underbrace{\sum_{(i,j)\in E} u_i^{(i,j)}(a_i^*, \bar{a}_j) - \Delta(T)}_{RHS}$$

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2. Summing the LHS over all players:

$$LHS = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{(i,j)\in E} u_i^{(i,j)}(a_i^t, a_j^t) = \frac{1}{T}\sum_{t=1}^{T} 0 = 0$$

(why?)

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2. Now summing the RHS:

$$RHS = \sum_{i=1}^{n}\sum_{(i,j)\in E} u_i^{(i,j)}(a_i^*, \bar{a}_j) - n \cdot \Delta(T)$$

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4. Lets think about each term...

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2. So for each player i:

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   (why?)
3. Tada!

Thanks!

See you next class — stay healthy!