# The Polynomial Weights Algorithm

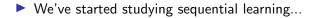
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February 4 2025

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## Overview

We've started studying sequential learning...

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What do we do without that assumption?

The Setting:

▶ There are *N* experts who will make predictions in *T* rounds.

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- Easy Case: there is one *perfect* expert who never makes a mistake (but we don't know who he is).

Algorithm 1 The Halving Algorithm

Let  $S^1 \leftarrow \{1, \ldots, N\}$  be the set of all experts. for t = 1 to T do Let  $S_U^t = \{i \in S : p_i^t = U\}$  be the set of experts in  $S^t$  who predict up, and  $S_D^t = S^t \setminus S_U^t$  be the set who predict down. Predict with the majority vote: If  $|S_U^t| > |S_D^t|$ , predict  $p_A^t = U$ , else predict  $p_A^t = D$ . Eliminate all experts that made a mistake: If  $o^T = U$ , then let  $S^{t+1} = S_U^t$ , else let  $S^{t+1} = S_D^t$ end for

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### Proof.

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- 4. Hence  $|S^t| \ge 1$  for all t.

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- 3. On the other hand, the perfect expert is never eliminated.
- 4. Hence  $|S^t| \ge 1$  for all t.
- 5. Since  $|S^1| = N$ , this means there can be at most log N mistakes.

#### Algorithm 2 The Iterated Halving Algorithm

Let  $S^1 \leftarrow \{1, \ldots, N\}$  be the set of all experts. for t = 1 to T do If  $|S^t| = 0$  Reset: Set  $S^t \leftarrow \{1, \ldots, N\}$ . Let  $S_U^t = \{i \in S : p_i^t = U\}$  be the set of experts in  $S^t$  who predict up, and  $S_D^t = S^t \setminus S_U^t$  be the set who predict down. Predict with the majority vote: If  $|S_U^t| > |S_D^t|$ , predict  $p_A^t = U$ , else predict  $p_A^t = D$ . Eliminate all experts that made a mistake: If  $o^T = U$ , then let  $S^{t+1} = S_U^t$ , else let  $S^{t+1} = S_D^t$ end for

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- 4. in particular, between any two resets, the *best* expert has made at least 1 mistake.
- 5. This gives the claimed bound.

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3. What should we do instead?

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- 3. What should we do instead?
- 4. How about just downweight experts who make mistakes?

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# The Weighted Majority Algorithm

Algorithm 3 The Weighted Majority Algorithm

Set weights  $w_i^1 \leftarrow 1$  for all experts *i*. for t = 1 to *T* do Let  $W_U^t = \sum_{i:p_i^t=U} w_i$  be the weight of experts who predict up, and  $W_D^t = \sum_{i:p_i^t=D} w_i$  be the weight of those who predict down. Predict with the weighted majority vote: If  $W_U^t > W_D^t$ , predict  $p_A^t = U$ , else predict  $p_A^t = D$ . Down-weight experts who made mistakes: For all *i* such that  $p_i^t \neq o^t$ , set  $w_i^{t+1} \leftarrow w_i^t/2$ end for

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Note that log(N) is a fixed constant, so the ratio of mistakes the algorithm makes compared to OPT is just 2.4 in the limit – not great, but not bad.

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- 7. Together we have:

$$\begin{split} \left(\frac{1}{2}\right)^{\text{OPT}} &\leq W \leq N \left(\frac{3}{4}\right)^{M} \\ & \left(\frac{4}{3}\right)^{M} \leq N \cdot 2^{\text{OPT}} \\ & M \leq 2.4(\text{OPT} + \log(N)) \end{split}$$

We've been doing well! What do we want in an algorithm?

1. It to make only 1 times as many mistakes as the best expert in the limit, rather than 2.4 times...

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- 2. It to be able to handle *N* distinct actions (a separate action for each expert), not just two (up and down)...
- 3. It to be able to handle experts having arbitrary costs in [0,1] at each round, not just binary costs (right vs. wrong)

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- 3. The total loss of expert *i* is  $L_i^T = \sum_{t=1}^T \ell_i^t$ , and the total loss of the algorithm is  $L_A^T = \sum_{t=1}^T \ell_A^t$ .

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 Is randomized — chooses which expert to follow with probability proportional to its weight.

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Theorem For any sequence of losses, and any expert k:  $\frac{1}{T} \mathbb{E}[L_{PW}^{T}] \leq \frac{1}{T} L_{k}^{T} + \epsilon + \frac{\ln(N)}{\epsilon \cdot T}.$  In particular, setting  $\epsilon = \sqrt{\frac{\ln(N)}{T}}:$ 

$$\frac{1}{T} \mathbf{E}[L_{PW}^{T}] \leq \frac{1}{T} \min_{k} L_{k}^{T} + 2\sqrt{\frac{\ln(N)}{T}}$$

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- 1. The average loss of the algorithm quickly approaches the average loss of the best expert exactly, at a rate of  $1/\sqrt{T}$ .
- 2. This works against an *arbitrary* sequence of losses, which might be chosen adaptively by an adversary.

Theorem For any sequence of losses, and any expert k:  $\frac{1}{T} \mathbb{E}[L_{PW}^{T}] \leq \frac{1}{T} L_{k}^{T} + \epsilon + \frac{\ln(N)}{\epsilon \cdot T}.$  In particular, setting  $\epsilon = \sqrt{\frac{\ln(N)}{T}}:$ 

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$$\operatorname{E}[L_{PW}^{T}] = \sum_{t=1}^{T} F^{t}$$
.

3. We also know:

$$F^t = \frac{\sum_{i=1}^N w_i^t \ell_i^t}{W^t}$$

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5. So by induction:

$$W^{T+1} = N \prod_{t=1}^{T} (1 - \epsilon F^t)$$

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3. Fin.

# Thanks!

See you next class!

