Congestion Games

Aaron Roth

University of Pennsylvania

January 23 2025

Overview

- Today we'll give study a structured class of large games.

Overview

- Today we'll give study a structured class of large games.
- We'll study a simple, natural dynamic, and show it converges to Nash equilibrium.

Overview

- Today we'll give study a structured class of large games.
- We'll study a simple, natural dynamic, and show it converges to Nash equilibrium.
- Our first "computationally plausible" set of predictions in a large interaction.

Large Games

Q: How many numbers do we need to write down to represent an n player k action game?

Large Games

Q: How many numbers do we need to write down to represent an n player k action game?
A: We need k^n numbers just to encode a single utility function.

Large Games

Q: How many numbers do we need to write down to represent an n player k action game?
A: We need k^n numbers just to encode a single utility function.

Unreasonable to expect anyone to understand such an object.

Large Games

Q: How many numbers do we need to write down to represent an n player k action game?
A: We need k^n numbers just to encode a single utility function.

Unreasonable to expect anyone to understand such an object.
So: we need to think about structured, concisely defined games.

Example 1: Traffic Routing

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Definition

A congestion game is defined by:

1. A set of n players P

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Definition

A congestion game is defined by:

1. A set of n players P
2. A set of m facilities F

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Definition

A congestion game is defined by:

1. A set of n players P
2. A set of m facilities F
3. For each player i, a set of actions A_i. Each action $a_i \in A_i$ represents a subset of the facilities: $a_i \subseteq F$.

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Definition

A congestion game is defined by:

1. A set of n players P
2. A set of m facilities F
3. For each player i, a set of actions A_i. Each action $a_i \in A_i$ represents a subset of the facilities: $a_i \subseteq F$.
4. For each facility $j \in F$, a cost function $\ell_j : \{0, \ldots, n\} \to \mathbb{R}_{\geq 0}$. $\ell_j(k)$ represents "the cost of facility j when k players are using it".

Congestion Games

Convention: Players have *cost functions* they want to minimize rather than *utility functions* they want to maximize.

Definition

A congestion game is defined by:

1. A set of n players P
2. A set of m facilities F
3. For each player i, a set of actions A_i. Each action $a_i \in A_i$ represents a subset of the facilities: $a_i \subseteq F$.
4. For each facility $j \in F$, a cost function $\ell_j : \{0, \ldots, n\} \rightarrow \mathbb{R}_{\geq 0}$. $\ell_j(k)$ represents "the cost of facility j when k players are using it".

Player costs are then defined as follows. For action profile $a = (a_1, \ldots, a_n)$ define $n_j(a) = |\{i : j \in a_i\}|$ to be the number of players using facility j. Then the cost of agent i is:

$$c_i(a) = \sum_{j \in a_i} \ell_j(n_j(a))$$

Example 2: Network Creation

Congestion games

- Ok, so we can concisely describe a large game...

Congestion games

- Ok, so we can concisely describe a large game...
- So what? What can we do with it?

Congestion games

- Ok, so we can concisely describe a large game...
- So what? What can we do with it?
- Do they have pure strategy Nash equilibria?

Congestion games

- Ok, so we can concisely describe a large game...
- So what? What can we do with it?
- Do they have pure strategy Nash equilibria?
- Can computationally bounded, uncoordinated players find one?

Congestion games

- Ok, so we can concisely describe a large game...
- So what? What can we do with it?
- Do they have pure strategy Nash equilibria?
- Can computationally bounded, uncoordinated players find one?
- i.e. are pure strategy Nash equilibria computationally plausible predictions?

Congestion games

- Ok, so we can concisely describe a large game...
- So what? What can we do with it?
- Do they have pure strategy Nash equilibria?
- Can computationally bounded, uncoordinated players find one?
- i.e. are pure strategy Nash equilibria computationally plausible predictions?
- Lets study a simple dynamic...

Best (Better) Response Dynamics

The basic idea:

Best (Better) Response Dynamics

The basic idea:

1. Players start playing *arbitrary* actions.

Best (Better) Response Dynamics

The basic idea:

1. Players start playing *arbitrary* actions.
2. In arbitrary order, players take turns changing their action if doing so can improve their utility.

Best (Better) Response Dynamics

The basic idea:

1. Players start playing *arbitrary* actions.
2. In arbitrary order, players take turns changing their action if doing so can improve their utility.
3. Forever...

Best (Better) Response Dynamics

Algorithm 1 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Let a'_i be such that $c_i(a'_i, a_{-i}) < c(a)$.
  Set $a_i = a'_i$.
end while
Halt and return a.

Best (Better) Response Dynamics

Algorithm 2 Best Response Dynamics

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Let a'_i be such that $c_i(a'_i, a_{-i}) < c(a)$.
  Set $a_i = a'_i$.
end while
Halt and return a.

Claim

If best response dynamics halts, it returns a pure strategy Nash equilibrium.

Best (Better) Response Dynamics

Algorithm 3 Best Response Dynamics

Initialize $a = (a_1, \dots, a_n)$ to be an arbitrary action profile.
while There exists i such that $a_i \notin \arg\min_{a \in A_i} c_i(a, a_{-i})$ **do**
  Let a'_i be such that $c_i(a'_i, a_{-i}) < c(a)$.
  Set $a_i = a'_i$.
end while
Halt and return a.

Claim

If best response dynamics halts, it returns a pure strategy Nash equilibrium.

Proof.

Immediate from halting condition – by definition, every player must be playing a best response. □

Best (Better) Response Dynamics

Does best response dynamics always halt?

Best (Better) Response Dynamics

Does best response dynamics always halt?

No: Consider matching pennies/Rock Paper Scissors.

Best (Better) Response Dynamics

Does best response dynamics always halt?
No: Consider matching pennies/Rock Paper Scissors.

Theorem

Best response dynamics always halt in congestion games.

Best (Better) Response Dynamics

Does best response dynamics always halt?
No: Consider matching pennies/Rock Paper Scissors.

Theorem

Best response dynamics always halt in congestion games.

Corollary

All congestion games have at least one pure strategy Nash equilibrium.

Analysis of BRD in Congestion Games

1. Consider the *potential function* $\phi : A \to \mathbb{R}$:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

(Note: *not* social welfare)

Analysis of BRD in Congestion Games

1. Consider the *potential function* $\phi : A \to \mathbb{R}$:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

   (Note: *not* social welfare)

2. How does ϕ change in one round of BRD? Say i switches from a_i to $b_i \in A_i$.

Analysis of BRD in Congestion Games

1. Consider the *potential function* $\phi : A \to \mathbb{R}$:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

   (Note: *not* social welfare)

2. How does ϕ change in one round of BRD? Say i switches from a_i to $b_i \in A_i$.
3. Well... We know it must have decreased player i's cost:

$$\begin{aligned}\Delta c_i &\equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &< 0\end{aligned}$$

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

1. The change in potential is:

$$\begin{aligned} \Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &= \Delta c_i \end{aligned}$$

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

1. The change in potential is:

$$\begin{aligned} \Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &= \Delta c_i \end{aligned}$$

2. Therefore, the change in potential is strictly *negative*

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

1. The change in potential is:

$$\begin{aligned}\Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &= \Delta c_i\end{aligned}$$

2. Therefore, the change in potential is strictly *negative*
3. So... since ϕ can take on only finitely many values, this cannot go on forever.

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

1. The change in potential is:

$$\begin{aligned} \Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &= \Delta c_i \end{aligned}$$

2. Therefore, the change in potential is strictly *negative*
3. So... since ϕ can take on only finitely many values, this cannot go on forever.
4. And hence BRD halts in congestion games...

Analysis of BRD in Congestion Games

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

1. The change in potential is:

$$\begin{aligned} \Delta\phi &\equiv \phi(b_i, a_{-i}) - \phi(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a)+1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) \\ &= \Delta c_i \end{aligned}$$

2. Therefore, the change in potential is strictly *negative*
3. So... since ϕ can take on only finitely many values, this cannot go on forever.
4. And hence BRD halts in congestion games...
5. Which proves the *existence* of pure strategy Nash equilibria!

Efficiency

But... How long does it take?

Efficiency

But... How long does it take?

Our proof gives only an exponential convergence bound... And it might really take that long!

Efficiency

But... How long does it take?

Our proof gives only an exponential convergence bound... And it might really take that long!

Lets consider approximation...

Approximation

Definition

An action profile $a \in A$ is an ϵ-approximate pure strategy Nash equilibrium if for every player i, and for every action $a'_i \in A_i$:

$$c_i(a_i, a_{-i}) \leq c_i(a'_i, a_{-i}) + \epsilon$$

i.e. nobody can gain more than ϵ by deviating.

.

Approximate Best Response Dynamics

Algorithm 4 FindApproxNash(ϵ)

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i, a'_i such that $c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
end while
Halt and return a.

Approximate Best Response Dynamics

Algorithm 5 FindApproxNash(ϵ)

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i, a_i' such that $c_i(a_i', a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
end while
Halt and return a.

Claim

If FindApproxNash(ϵ) halts, it returns an ϵ-approximate pure strategy Nash equilibrium

Approximate Best Response Dynamics

Algorithm 6 FindApproxNash(ϵ)

Initialize $a = (a_1, \ldots, a_n)$ to be an arbitrary action profile.
while There exists i, a'_i such that $c_i(a'_i, a_{-i}) \leq c_i(a_i, a_{-i}) - \epsilon$ **do**
  Set $a_i = \arg\min_{a \in A_i} c_i(a, a_{-i})$
end while
Halt and return a.

Claim

If FindApproxNash(ϵ) halts, it returns an ϵ-approximate pure strategy Nash equilibrium

Proof.

Immediately, by definition. □

Analysis

Theorem

In any congestion game, FindApproxNash(ϵ) halts after at most:

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

steps, where $c_{max} = \max_{j,k} \ell_j(k)$ *is the maximum facility cost.*

Analysis

Theorem

In any congestion game, FindApproxNash(ϵ) halts after at most:

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

steps, where $c_{max} = \max_{j,k} \ell_j(k)$ *is the maximum facility cost.*

Proof.

We revisit the potential function ϕ. Recall that $\Delta c_i = \Delta\phi$ on any round when player i moves.

Analysis

Theorem

In any congestion game, FindApproxNash(ϵ) halts after at most:

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

steps, where $c_{max} = \max_{j,k} \ell_j(k)$ is the maximum facility cost.

Proof.

We revisit the potential function ϕ. Recall that $\Delta c_i = \Delta \phi$ on any round when player i moves.

Observe also that at every round, $\phi \geq 0$, and

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k) \leq n \cdot m \cdot c_{max}$$

Analysis

Theorem

In any congestion game, FindApproxNash(ϵ) halts after at most:

$$\frac{n \cdot m \cdot c_{max}}{\epsilon}$$

steps, where $c_{max} = \max_{j,k} \ell_j(k)$ is the maximum facility cost.

Proof.

We revisit the potential function ϕ. Recall that $\Delta c_i = \Delta\phi$ on any round when player i moves.

Observe also that at every round, $\phi \geq 0$, and

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k) \leq n \cdot m \cdot c_{max}$$

By definition of the algorithm, we have $\Delta c_i = \Delta\phi \leq -\epsilon$ at every round, and so the theorem follows. □

Thanks!

See you next class!