Basic Definitions

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1. Average guess: 23.71

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- 2. 2/3 the average: 17.06

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- 3. Closest Guess: 15.0000069693489

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4. Winner: Forest James Ho-Chen

1. Guesses above 66.66: 7



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- 2. Guesses above 44.44: 8

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- 1. Guesses above 66.66: 7
- 2. Guesses above 44.44: 8
- 3. Guesses above 29.33: 10

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- 1. Guesses above 66.66: 7
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- 3. Guesses above 29.33: 10
- 4. Guesses above 19.56: 15

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- 1. Guesses above 66.66: 7
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- 3. Guesses above 29.33: 10
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- 5. Guesses above 13.17: 18

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7. ... Guesses of 0: 4

Overview

Today we'll give (review) the basic definitions that will underly our study this semester.

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Solution concepts: Nash equilibrium

A Game

Definition

A game is an interaction defined by:

► A set of players P



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A finite set of actions A_i for each player i ∈ P. We write A = ×ⁿ_{i=1}A_i to denote the action space for all players, and A_{-i} = ×_{j≠i}A_j to denote the action space of all players excluding player j.

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• A utility function $u_i : A \to \mathbb{R}$ for each player $i \in P$.

Basic assumption: players will always try and act so as to maximize their utility.

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Definition

The *best-response* to a set of actions $a_{-i} \in A_{-i}$ for a player *i* is any action $a_i \in A_i$ that maximizes $u_i(a_i, a_{-i})$:

$$a_i \in rg\max_{a \in A_i} u_i(a, a_{-i})$$

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Interlude

Question: Is game theory just for sociopaths?



Interlude

Question: Is game theory just for sociopaths? **Answer:** Not necessarily. (Assumes only that people have consistent preferences)

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The General Idea for Prediction

"In any stable situation, all players should be playing a best response."

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"In any stable situation, all players should be playing a best response."

(Otherwise, by definition, the situation would not be stable – somebody would want to change their action.)

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When are there stable solutions?

Definition

For a player *i*, an action $a \in A_i$ (weakly) dominates action $a' \in A_i$ if it is always beneficial to play *a* over *a'*. That is, if for all $a_{-i} \in A_{-i}$:

$$u_i(a,a_{-i}) \geq u_i(a',a_{-i})$$

and the inequality is strict for some $a_{-i} \in A_{-i}$.

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and the inequality is strict for some $a_{-i} \in A_{-i}$.

Can normally eliminate dominated strategies from consideration – there is never a situation in which they are the (unique) best response.

Dominant Strategies

Definition

An action $a \in A_i$ is *dominant* for player *i* if it weakly dominates all actions $a' \neq a \in A_i$.

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1. A very strong guarantee – Always a best response.

Dominant Strategies

Definition

An action $a \in A_i$ is *dominant* for player *i* if it weakly dominates all actions $a' \neq a \in A_i$.

- 1. A very strong guarantee Always a best response.
- 2. No need to reason about what your opponents are doing.

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Dominant Strategy Equilibrium

Dominant strategies normally don't exist, but when they do, predictions are easy.

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Definition

An action profile $a = (a_1, ..., a_n) \in A$ is a *dominant strategy* equilibrium of the game $(P, \{A_i\}, \{u_i\})$ if for every $i \in P$, a_i is a dominant strategy for player *i*.

Example: Prisoner's Dilemma

	Confess	Silent
Confess	(1, 1)	(5,0)
Silent	(0,5)	(3,3)

Figure: Prisoner's Dilemma



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(Confess, Confess) is a dominant strategy equilibrium is Prisoner's Dilemma.

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What if there are no dominant strategies?

It still makes sense to eliminate *dominated* strategies from consideration.

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Sometimes, once you've done this, new strategies have become dominated. What if there are no dominant strategies?

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- ▶ We can consider eliminating dominated strategies *iteratively*.

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What if there are no dominant strategies?

- It still makes sense to eliminate *dominated* strategies from consideration.
- Sometimes, once you've done this, new strategies have become dominated.
- ▶ We can consider eliminating dominated strategies *iteratively*.
- If we are lucky, "iterated elimination of dominated strategies" leads to a unique surviving strategy profile.

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Iterated Elimination: Example 1

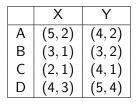


Figure: Example 1

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Iterated Elimination: Example 2

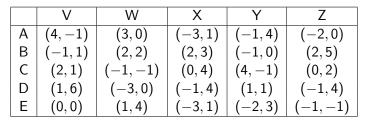


Figure: Example 2

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We can still ask for a "stable" profile of actions.

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Definition

A profile of actions $a = (a_1, ..., a_n) \in A$ is a *pure strategy Nash* Equilibrium if for each player $i \in P$ and for all $a'_i \in A_i$:

$$u_i(a_i,a_{-i}) \geq u_i(a_i',a_{-i})$$

i.e. simultaneously, all players are playing a best response to one another.

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Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

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Claim

If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium.

Proof. Homework!

Problem 1: They don't always exist.

	Heads	Tails
Heads	(1, -1)	(-1, 1)
Tails	(-1, 1)	(1, -1)

Figure: Matching Pennies

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Problem 2: They aren't always unique.

	Bach	Stravinsky
Bach	(5,1)	(0,0)
Stravinsky	(0,0)	(1, 5)

Figure: Bach of Stravinsky

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A two-player game is *zero-sum* if for all $a \in A$, $u_1(a) = -u_2(a)$. (i.e. the utilities of of both players sum to zero at every action profile)

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- 1. e.g. Matching Pennies.
- 2. In matching pennies you should randomize to thwart your opponent: Flip a coin and play heads 50% of the time, and tails 50% of the time.

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Definition

A mixed-strategy $p_i \in \Delta A_i$ is a probability distribution over actions $a_i \in A_i$: i.e. a set of numbers $p_i(a_i)$ such that:

1.
$$p_i(a_i) \ge 0$$
 for all $a_i \in A_i$

2.
$$\sum_{a_i\in A_i}p_i(a_i)=1.$$

For $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$, we write:

$$u_i(p) = E_{a_i \sim p_i}[u_i(a)]$$

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Mixed Strategy Nash Equilibria

Definition

A *mixed strategy Nash equilibrium* is a tuple

 $p = (p_1, \ldots, p_n) \in \Delta A_1 \times \ldots \times \Delta A_n$ such that for all *i*, and for all $a_i \in A_i$:

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A mixed strategy Nash equilibrium is a tuple

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Theorem (Nash)

Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.

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Theorem (Nash)

Every game with a finite set of players and actions has a mixed strategy Nash equilibrium.

But... The proof is non-constructive, so its not necessarily clear how to find one of these, even though they exist

Thanks!

See you next class!

