Posted Pricings and Prophet Inequalities

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- But auctions are difficult to run. They require e.g. all participants to be present and coordinating.
- Many things are instead sold via posted prices.
- This lecture: How to approximate the welfare and revenue of the optimal auction with posted prices

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Are there choices of p_i that allow us to approximate the welfare or revenue of the optimal auction?

Prophets and Profits (an Interlude)

Consider the following game:

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- How well can you do?

The Prophet Inequality

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Theorem

For every set of distributions $G_1, \ldots, G_n$, there is a threshold strategy that guarantees reward at least $\frac{1}{2}\mathrm{E}[\max_i \pi_i]$.

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- We'll prove the prophet inequality by decomposing expected reward between:
  1. Expected revenue, and
  2. Expected buyer utility.

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- Welfare = Revenue + Buyer Utility.
- Strategy: Prove lower bounds on expected revenue and buyer utility separately.

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Immediate implications for welfare maximization!

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- What about for revenue?

Revenue

Recall that for monotone allocation rules X paired with truthful pricings P:

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- We need to use different prices for different types of bidders, but approximate optimal revenue.

Thanks!

See you next class — stay healthy!