Maximizing Revenue in Expectation

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March 26 2025

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- But many auctions have a more pecuniary goal. What if we want to maximize revenue?
- What does that mean? What is our benchmark?
- This lecture: a case study for single item auctions.

Reasonable Benchmarks?

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- What about for revenue? Not so simple.
- Consider a single bidder, single item auction. Offering a fixed price p is always dominant strategy truthful.
- Revenue is p if $v_i \geq p$, 0 otherwise.
- So ex-post, the revenue-optimal auction sets $p = v_i$... But ex-ante, we don't have enough information.

The Average Case

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Where $F(p) = \Pr_{v \sim D}[v \leq p]$.

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  Where $F(p) = \Pr_{v \sim D}[v \leq p]$.
- E.g. if D is uniform on $[0, 1]$, then $F(p) = p$ and:

$$\max_p Rev(p) = \frac{1}{2} \cdot (1 - \frac{1}{2}) = \frac{1}{4}$$

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- And we know:

$$P_i(v) = v_i \cdot X_i(v) - \int_0^{v_i} X_i(z, v_{-i})dz$$

Myserson Optimal Auctions

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- Notation: $f(p)$ is the pdf of D.

$$F(p) = \Pr_{v \sim D}[v \leq p] = \int_0^p f(v)dv$$

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So: We want to maximize

$$\mathrm{E}_{v \sim D^n}\left[\sum_{i=1}^{n} \phi(v_i) \cdot X(v)\right] \qquad \underbrace{\phi(v_i) = \left(v_i - \frac{(1 - F(v_i))}{f(v_i)}\right)}_{\text{"Virtual Value"}}$$

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- Our objective looks just like welfare with values replaced by virtual values.
- (Pointwise) optimal allocation rule: Give the item to the bidder i with highest $\phi(v_i)$ if it's positive. Otherwise give the item to nobody.
- This is a monotone allocation rule if D is *regular*: $\phi(v_i)$ is monotone.
  - e.g. if D is uniform, $\phi(v_i) = v_i - (1 - v_i) = 2v_i - 1$
  - Note that $\phi^{-1}(0)$ recovers the optimal $p = 1/2$ for a single bidder.

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What do revenue maximizing auctions look like? (when v_i drawn iid from regular D)

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- Remarkable — Simple eBay style auction is *the best possible*.

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- Doesn't extend beyond single parameter domains...
- Requires knowledge of D...

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- What about a Vickrey auction with 2 bidders?
- $$Rev(VA) = \mathrm{E}_{v_1, v_2 \sim D}[\min(v_1, v_2)] = 1/3$$

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- What about a Vickrey auction with 2 bidders?
- $$Rev(VA) = \mathrm{E}_{v_1, v_2 \sim D}[\min(v_1, v_2)] = 1/3$$
- So we might be better off maximizing welfare with more bidders...

The Bulow/Klemperer Theorem

Theorem

Consider bidders drawn i.i.d. from a regular distribution D. For any $n \geq 1$, the Vickrey auction with $n+1$ bidders has higher expected revenue than the revenue optimal auction with n bidders.

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So recruiting just *one* extra bidder is worth more than optimizing revenue for the current population.

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Observations:

1. The revenue of A is exactly equal to the optimal revenue obtainable from n bidders.
2. A *always* allocates the item.

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- Since D is regular, ϕ is monotone: this is $\arg\max_i v_i$ — the Vickrey allocation!
- So: The Vickrey-auction with $n+1$ bidders has only higher revenue than the optimal n bidder auction.

Thanks!

See you next class — stay healthy!