Approximation in Mechanism Design

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2. This lecture: We grapple with computational complexity.
3. Recall the VCG mechanism must solve:

$$X(v) = \arg\max_{a \in A} \sum_i v_i(a)$$

4. What do we do when this problem is hard to solve – e.g. NP-complete?

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3. If we only find an alternative that achieves 99% of the optimal welfare, these guarantees break.
4. As a case study, we will consider *Knapsack auctions*.

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Definition

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- Each bidder $i \in \{1, \ldots, n\}$ has a public *size* $w_i \in \mathbb{R}_{\geq 0}$.

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- For each $a \in A$ we write $a_i = 1$ if $i \in a$.
- These are single parameter domains. Each bidder i has a real value $v_i \in \mathbb{R}_{\geq 0}$, and their value for alternative a is $v_i \cdot a_i$

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- So: likely no polynomial time algorithm for this task.
- A natural problem, modelling e.g. selling seats on an airplane to people who have different sized parties.

Knapsack Auctions

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So what should we do?

- We could find a choice rule which *approximates* the social welfare objective and a pricing rule which makes it dominant strategy truthful.
- We know that the only way to do this we have to find a *monotone non-decreasing* approximation algorithm.

Approximation

Definition

For a set of values and weights $v, w \in \mathbb{R}^n_{\geq 0}$, let:

$$\mathrm{OPT}(v, w) = \max_{S \subseteq [n] : \sum_{i \in S} w_i \leq B} \sum_{i \in S} v_i$$

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A is an α-approximation algorithm for the Knapsack problem if for every $v, w \in \mathbb{R}^n_{\geq 0}$, $A(v,w) = S$ such that:

1. S is a feasible solution: $\sum_{i \in S} w_i \leq B$
2. S approximates OPT to within a factor of α: $\sum_{i \in S} v_i \geq \frac{\mathrm{OPT}(v,w)}{\alpha}$

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Monotone Non Decreasing: for every $v, w \in \mathbb{R}^n_{\geq 0}$, and for every i and $v'_i > v_i$, if $S = A(v, w)$ and $S' = A((v'_i, v_{-i}), w)$, then:

$$i \in S \Rightarrow i \in S'.$$

Our Goal

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- Goal: Come up with a monotone algorithm A that is also an α-approximation algorithm for the Knapsack problem.
- Observation: We can write the knapsack problem in the following integer linear optimization form:

$$\text{maximize} \sum_{i=1}^{n} x_i \cdot v_i$$

$$\text{such that:}$$

$$\sum_{i=1}^{n} x_i \cdot w_i \leq B$$

$$x_i \in \{0, 1\} \quad \forall i$$

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- We are unlikely to be able to reason about structure of the optimal solution.
- Instead, consider the following "relaxed" problem in which the x_i can be fractional:

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$$\text{such that:}$$

$$\sum_{i=1}^{n} x_i \cdot w_i \leq B$$

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The Fractional Problem

- Write $\mathrm{OPT}_F(v, w)$ for the optimal value of this “fractional” problem.

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Lemma

For all $v, w \in \mathbb{R}^n_{\geq 0}$:

$$\mathrm{OPT}_F(v, w) \geq \mathrm{OPT}(v, w)$$

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Proof.

Any optimal solution to the integer version of the problem is a *feasible* solution to the fractional version, so $\mathrm{OPT}_F(v, w) \geq \mathrm{OPT}(v, w)$ □

Understanding the Fractional Problem

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- The fractional relaxation is simpler/easier to understand:

Lemma

Let x be a fractional solution obtaining value $\mathrm{OPT}_F(v, w)$ in the fractional knapsack problem. Let $i, j \in [n]$ be any pair of agents such that:

$$\frac{v_i}{w_i} > \frac{v_j}{w_j}.$$

Then $x_j > 0 \rightarrow x_i = 1$

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- Plan: Define a new solution x' and argue that it:
    1. Is feasible, and
    2. Has higher objective value, contradicting the optimality of x.

Understanding the Fractional Problem

- Let $x'_\ell = x_\ell$ for all $\ell \notin \{i, j\}$, and let

  $$x'_j = x_j - \delta$$

  and

  $$x'_i = x_i + \delta \frac{w_j}{w_i}.$$

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- Note that x' continues to satisfy the knapsack constraint: the change in size of the bundle was:

$$(\delta \frac{w_j}{w_i}) \cdot w_i - \delta w_j = \delta w_j - \delta w_j = 0$$

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- Hence (because x was feasible) x' is feasible.

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- This contradicts the optimality of x.

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We can now give a simple combinatorial algorithm for the fractional version of the knapsack problem.
Given our lemma, we know this algorithm must be optimal.
FractionalKnapsack(v, w):

```
Sort bidders in decreasing order by $\frac{v_i}{w_i}$ and set size $\leftarrow 0$ and $i \leftarrow 1$.
while size$+w_i \leq B$ do
    Set $x_i \leftarrow 1$, $\mathrm{size} \leftarrow \mathrm{size} + w_i$, $i \leftarrow i + 1$.
end while
Set $x_i \leftarrow \frac{B - \mathrm{size}}{w_i}$ and Set $x_j = 0$ for all $j > i$.
Return $x$.
```

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- What if we just remove the last step? How does this do?
- Terribly! Consider the following example.

Example

We have two agents with $w_1 = v_1 = 10$ and $w_2 = 1$ and $v_2 = 1.1$. $B = 10$. Note that $\mathrm{OPT}(v, w) = 10$ However, $v_2/w_2 > v_1/w_1$, so the algorithm first picks agent 2, and then has no remaining space for agent 1. So the algorithm's solution has value only 1.1. We could extend this example to make the algorithm's solution arbitrarily worse!

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- Lets try again. Note that WLOG, we can assume that for all i, $w_i \leq B$.
  Greedy2(v, w):

```
Sort bidders in decreasing order by $\frac{v_i}{w_i}$ and set size $\leftarrow 0$ and
$i \leftarrow 1$. Set $S \leftarrow \emptyset$.
while size$+w_i \leq B$ do
    Set $S \leftarrow S \cup \{i\}$, $\mathrm{size} \leftarrow \mathrm{size} + w_i$, $i \leftarrow i + 1$.
end while
if $\sum_{j \in S} v_j \geq v_i$ then
    Output $S$.
else
    Output $\{i^*\}$ where $i^* = \arg\max_i v_i$.
end if
```

The Integer Problem

Theorem

Greedy2 achieves a 2-approximation algorithm for the Knapsack problem.

The Integer Problem

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$$j \notin S \cup \{i\} \Rightarrow x_j^* = 0$$

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- Hence:

$$\sum_{j \in S} v_j + v_i \geq \mathrm{OPT}_F(v, w) \geq \mathrm{OPT}(v, w).$$

- Therefore:

$$\max(\sum_{j \in S} v_j, v_i) \geq \frac{OPT(v, w)}{2}$$

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- And $v_{i^*} \geq v_i$ by definition. So:

$$\max(\sum_{i \in S} v_i, v_{i^*}) \geq \frac{OPT(v, w)}{2}$$

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Theorem

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Hence, there is a dominant strategy truthful 2-approximation algorithm for the Knapsack Auction problem.

Establishing Truthfulness

- Fix any $w, v \in \mathbb{R}^n_{\geq 0}$, any agent i, and let $v'_i > v_i$. Write $v' = (v'_i, v_{-i})$. Let $T = Greedy2(v, w)$ and $T' = Greedy2(v', w)$.

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- Write $S \doteq S(v, w)$ and $S' \doteq S(v', w)$ for the intermediate sets S generated by Greedy2 on each instance.

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- To show: $i \in T \Rightarrow i \in T'$.
- Write $S \doteq S(v, w)$ and $S' \doteq S(v', w)$ for the intermediate sets S generated by Greedy2 on each instance.
- First we argue:

$$i \in S \Rightarrow i \in S'$$

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- So: if he was in the prefix S he is still in the prefix S'.
- Hence: If $T = S$ and $T' = S'$, then on this instance, the algorithm is monotone.

Establishing Truthfulness

- Note also that if $i \in S$, then $\sum_{j \in S'} v'_j \geq \sum_{j \in S} v_j$. Hence, if $i \in S$, then if $T = S$, $T' = S$.

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- The other case: $i = i^*$ and $v_i > \sum_{j \in S} v_j$.

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- The other case: $i = i^*$ and $v_i > \sum_{j \in S} v_j$.
- Here we also have $i \in T'$. i remains the highest bidder, and so is either output as $T' = \{i^*\}$ or is output as $T' = S'$ with $i \in S'$.

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- Here we also have $i \in T'$. i remains the highest bidder, and so is either output as $T' = \{i^*\}$ or is output as $T' = S'$ with $i \in S'$.
- So: we have shown that there exists a polynomial time 2-approximation for the Knapsack problem that makes truthful bidding a dominant strategy for all players.

Thanks!

See you next class