Auction Design in Single Parameter Domains

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- However, the VCG mechanism was particular to maximizing *social welfare*: $\sum_i v_i(a)$.
- What if we want to design an auction to maximize some other objective?

How far can we generalize?

One thing we can do is (slightly) generalize VCG to maximize any *affine* objective function:

$$\sum_{i=1}^{n} \alpha_i v_i(a) + \beta(a).$$

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You will prove this generalization on the homework.
What else can we do? In simple settings we can completely characterize the set of objective functions we can optimize truthfully.

Simple Settings

Definition (Single Parameter Domain)

A *single parameter domain* with a set of alternatives A is defined by a *public value summarization function*:

$$w_i : A \to \mathbb{R}$$

such that agent i's valuation function is parameterized by a real number $v_i \in \mathbb{R}$, and values outcome a at $v_i \cdot w_i(a)$

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i.e. single parameter domains are simple settings in which an agent's valuation can be described by a single real number representing her *relative preferences* over outcomes.

Examples

1. Single item auctions.

$$w_i(a) = \left\{ \begin{array}{ll} 1, & a = i; \\ 0, & \text{otherwise.} \end{array} \right.$$

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2. Buying a path in a network: agents are to edges in a network, experience cost if used. Mechanism would like to buy service from a set of agents that form a path, to optimize some objective. a is a set of edges and:

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3. Online Advertising: Each alternative a allocates a set of advertising slots. $a_{ij} = 1$ if slot j is allocated to advertiser i. Advertisers have utility v_i for each unique viewer. Let E_j be the set of viewers who see slot j. Here:

$$w_i(a) = \left| \bigcup_{j : x_{ij} = 1} E_j \right|$$

Key Concept: Monotone Choice Rules

Definition (Monotone Choice Rule)

A choice rule X for a single parameter domain is monotone-non-decreasing in v_i if for all $v_{-i} \in \mathbb{R}^{n-1}$, and for every $v_i' \geq v_i$:

$$w_i(X(v_i, v_{-i})) \leq w_i(X(v_i', v_{-i}))$$

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For example, in a single item auction: if an agent wins at bid v_i, he also wins at all bids $v_i' > v_i$.

Main Theorem

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Theorem

A mechanism defined in a single parameter domain can be made truthful if and only if $X(v)$ is monotone non-decreasing for all v_i. In this case, it can be made truthful by using payment rule:

$$P(v)_i = v_i w_i(a^*) - \int_0^{v_i} w_i(X(z, v_{-i}))dz$$

where $a^* = X(v)$.

Proof

Simpler notation: fix some agent i and v_{-i}, write v for v_i, and write $y(v)$ for $w(x(v))$.

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(i.e. in a single item auction, we now write $y(v) = 1$ if i is allocated at bid v, and 0 otherwise).
First the backwards direction: assuming $X(v)$ is monotone non-decreasing and the payment rule is as given, the auction is truthful.

Proof

To show: For all v':

$$v \cdot y(v) - P(v)_i \geq v \cdot y(v') - P(v')_i$$

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Plugging in the payment rule, this is:

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Which is equivalent to showing:

$$\int_0^v y(z)dz \geq \int_0^{v'} y(z)dz - (v' - v)y(v') \qquad (1)$$

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1. **Case 1**: $v' > v$. In this case, equation 1 becomes:

$$\int_v^{v'} y(z)dz \leq (v' - v)y(v')$$

But this is true by monotonicity. We know that $y(v') \geq y(z)$ for all $z \leq v'$, and so:

$$\int_v^{v'} y(z)dz \leq \int_v^{v'} y(v')dz = (v' - v)y(v')$$

(See Picture)

1. **Case 2**: $v' < v$. In this case, equation 1 becomes:

$$\int_{v'}^{v} y(z)dz \geq (v - v')y(v')$$

Again, this follows from monotonicity since we know that $y(v') \leq y(z)$ for all $z \geq v'$. Hence, we have:

$$\int_{v'}^{v} y(z)dz \geq \int_{v'}^{v} y(v')dz = (v - v')y(v')$$

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since a bidder with valuation v cannot benefit by misreporting value v'.
We also know that a bidder with valuation v' cannot benefit by misreporting v:

$$v' \cdot y(v') - P(v')_i \geq v' \cdot y(v) - P(v)_i$$

Proof

Adding these two inequalities, we get:

$$v \cdot y(v) + v' \cdot y(v') \geq v \cdot y(v') + v' \cdot y(v)$$

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Since $v' - v > 0$, we can divide to obtain:

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So the allocation rule must be monotone!

Thanks!

See you next class — stay healthy!