Auction Design

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Overview

Last lecture, we studied *pricing equilibria*. In this lecture, we continue our study of money as a means of exchange, from the perspective of mechanism design. Specifically, we begin our study of how to design *auctions*, which will be mechanisms for choosing outcomes, while managing the incentives of individuals to report to the mechanism their true preferences.

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- Agents have quasilinear utility functions. The utility that agent i experiences for outcome $o = (a, p)$ is:

$$u_i(o) = v_i(a) - p_i$$

Model

This could represent many things. e.g.

- A single item allocation problem. a represents *who* gets the good.
- A multi-item allocation problem. a represents a mapping from people to goods.
- A public goods problem. a represents whether or not a library is built.
- ...

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Lets lay out a "wish list" of desiderata that our dream auction would satisfy:

Desideratum 1: Safety

Definition (Individual Rationality)

A mechanism is individually rational (IR) if for every agent i and for every $v \in V^n$:

$$v_i(X(v)) \geq P(v)_i$$

i.e. nobody is ever asked to pay more than their (reported) value for the outcome.

Desideratum 2: Incentive Compatibility

Definition (Dominant Strategy Truthfulness)

A mechanism is *dominant strategy truthful* if for every agent i, for every $v \in V^n$, and for every alternative report $v_i' \in V$, we have:

$$u_i(X(v), P(v)) \geq u_i(X(v_i', v_{-i}), P(v_i', v_{-i}))$$

or equivalently:

$$v_i(X(v)) - P(v)_i \geq v_i(X(v_i', v_{-i})) - P(v_i', v_{-i})_i$$

Desideratum 3: Outcome Quality

Definition (Allocative Efficiency)

A mechanism is *allocatively efficient*, or "Social Welfare Maximizing", if for all $v \in V^n$, if $a = X(v)$, then for all $a' \in A$ we have:

$$\sum_i v_i(a) \geq \sum_i v_i(a')$$

Desideratum 4: Budget Balance

Definition (No Deficit)

A mechanism is *no deficit* if for all $v \in V^n$:

$$\sum_i P(v)_i \geq 0$$

i.e. in total, the mechanism does not have to pay to run the auction.

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So – can we satisfy all of our desiderata?

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2. We could try $p(v)_i = v_i$. Does this lead to an incentive compatible auction?
3. What about $p(v)_i = \arg\max_{j \neq X(v)} v_j$. Is this incentive compatible?

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- Note that its the same thing as the TV "English Auction"
- What about other pricing rules? What if the winner pays the 3rd highest price?
- Lets see if we can generalize this beyond single item auctions...

The Groves Mechanism

Definition

The *Groves Mechanism* has choice rule:

$$X(v) = \arg\max_{a \in A} \sum_i v_i(a)$$

and payment rule:

$$P(v)_i = h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*)$$

where h_i is an arbitrary function (crucially, independent of v_i), and $a^* = X(v)$ is the socially optimal outcome.

Note that the Groves mechanism is a family of mechanisms, instantiated by a choice of h_i.

Two Desiderata

Theorem

The Groves mechanism is dominant strategy incentive compatible and Allocatively efficient.

Proof.

It is allocatively efficient by definition, so it remains to verify that it is dominant strategy incentive compatible.

Two Desiderata

Proof.

Fix any agent i, and reports v_{-i} of the other players. We have:

$$u_i(X(v), P(v)) = v_i(a^*) + \sum_{j \neq i} v_j(a^*) - h_i(v_{-i})$$

where $a^* = \arg\max_{a \in A} \left(\sum_{j \neq i} v_i(a) + v_i'(a) \right)$. Agent i wishes to report v_i' to maximize his utility.

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But note that if agent i truthfully reports $v_i' = v_i$, then a^* maximizes this quantity by definition. Hence, it is a dominant strategy for all agents to report truthfully. □

Intuition

The payment scheme aligns the incentives of the agents and the mechanism designer: both prefer higher social welfare outcomes.

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- How can we pick h_i to achieve the no-deficit property without breaking individual rationality?

The VCG Mechanism

Definition (The Vickrey-Clarke-Groves (VCG) Mechanism)

The VCG mechanism is an instantiation of the Groves mechanism with

$$h_i(v_{-i}) = \sum_{j \neq i} v_j(a^*_{-i})$$

where $a^*_{-i} = \arg\max_{a \in A} \sum_{j \neq i} v_j(a)$ is the alternative that maximizes social welfare among all agents *other* than agent i. In other words, the VCG mechanism has payment rule:

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We will show that the VCG mechanism satisfies all of our desiderata.

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The VCG mechanism is allocatively efficient and dominant strategy incentive compatible.

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It is an instantiation of the Groves mechanism. □

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We need to show that Agent i's utility satisfies:

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But note that if this is not the case, since v_i is non-negative, we would have:

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But this would contradict the allocative efficiency of a^*!

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But note that this is always the case, since a^*_{-i} is explicitly defined to be the maximizer of $\sum_{j \neq i} v_j(a)$ over all $a \in A$. $\square$

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- So the VCG mechanism satisfies all of our wildest dreams, in an extremely general setting! Can end the class here?
- Not quite – we will see that the VCG mechanism still leaves a bit to be desired. It doesn't maximize other objectives (like e.g. revenue), and it isn't always computationally efficient.

Thanks!

See you next class — stay healthy!