Walrasian Equilibrium

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- An "exchange" and a "matching" problem.
- This lecture: We'll bring money into the picture in a matching like problem.
- And give a formalization of Adam Smith's "Invisible Hand"
- The thesis (in our simple model): simple, decentralized market dynamics lead to efficient outcomes.

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Buyers have *quasi-linear* utility functions: If each good $j \in G$ has a price p_j, then a buyer i gets utility for buying a bundle $S \subseteq G$:

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Questions: How we should *price* and *allocate* goods so that everyone is happy with their allocation. Is this even possible? If it is, can we do so and *also* achieve a high welfare allocation?

Some Definitions

First, feasibility:

Definition

An allocation $S_1, \ldots, S_n \subseteq G$ is *feasible* if for all $i \neq j$, $S_i \cap S_j = \emptyset$
We write OPT to denote the socially optimal feasible allocation:

$$\mathrm{OPT} = \max_{S_1,\ldots,S_n \text{ feasible}} \sum_i v_i(S)$$

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What is the right notion of equilibrium in a market?

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Definition

A set of prices p together with an allocation $S_1, \ldots, S_n$ form an (ϵ-approximate) *Walrasian equilibrium* if:

1. $S_1, \ldots, S_n$ is feasible, and
2. For all i, buyer i is receiving his (ϵ) most preferred bundle given the prices:

$$v_i(S_i) - \sum_{j \in S_i} p_j \geq \max_{S^* \subseteq G} \left(v_i(S^*) - \sum_{j \in S^*} p_j \right) - \epsilon$$

   and,
3. All unallocated items have zero price: for all $j \notin S_1 \cup \ldots \cup S_n$, $p_j = 0$.

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1. Do Walrasian equilibria always exist?

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2. If so, are they compatible with social welfare maximization?

The 2nd Question 1st

Theorem

If $S_1, \ldots, S_n$ form an ϵ-Walrasian equilibrium allocation, then they achieve nearly optimal welfare. In particular:

$$\sum_i v_i(S_i) \geq \mathrm{OPT} - \epsilon n$$

Proof

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4. Reordering:

$$\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S'_i) - \sum_{j \in S'_1 \cup \ldots \cup S'_n} p_j - \epsilon n$$

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$$\sum_i v_i(S_i) - \sum_{j \in S_1 \cup \ldots \cup S_n} p_j \geq \sum_i v_i(S_i') - \sum_{j \in S_1' \cup \ldots \cup S_n'} p_j - \epsilon n$$

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2. Rewriting:

$$\sum_i v_i(S_i) \geq \sum_i v_i(S_i') + (\sum_j p_j - \sum_{j \in S_1' \cup \ldots \cup S_n'} p_j) - \epsilon n \geq \sum_i v_i(S_i') - \epsilon n$$

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3. Finally, taking $S_1', \ldots, S_n'$ to be the optimal allocation gives the theorem. (Tada!)

Walrasian Equilibrium are Great! Do They Exist?

1. We'll start with a simple case: unit demand buyers (want to buy only 1 item):

$$v_i(S) = \max_{j \in S} v_i(\{j\})$$

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Theorem

For any set of unit demand buyers, a Walrasian equilibrium always exists.

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6. Deferred acceptance like...

Proof

Algorithm 1 The Ascending Price Auction with increment ϵ.

For all $j \in G$, set $p_j = 0$, $\mu(j) = \emptyset$.
while There exist any unmatched bidders **do**
  for Each unmatched bidder i **do**
    i "bids" on $j^* = \arg\max_j (v_{i,j} - p_j)$ if $v_{i,j^*} - p_{j^*} > 0$. Otherwise, bidder i drops out. (and is "matched" to nothing):
    $\mu(j^*)$ is now unmatched. Set $\mu(j^*) \leftarrow i$
    $p_{j^*} \leftarrow p_{j^*} + \epsilon$
  end for
end while
Output (p, μ).

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6. (Lemma Tada!)

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Lemma

The output (p, μ) of the ascending price auction is an ϵ-approximate Walrasian equilibrium.

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   4.2 For all $j \in S^*$, $\mu(j) \leftarrow i$, $p_j \leftarrow p_j + \epsilon/m$.

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3. So we do not want that when a bidder i bids, she abandons any of the goods she is currently matched to.
4. We can formalize this.

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1. For price vectors p, p', write $p \preceq p'$ to mean that $p_j \leq p'_j$ for all j. Let $w_i(p) = \arg\max_{S \subseteq G}(v_i(S) - \sum_{j \in S} p_j)$ be player i's demand set at prices p.

Definition

Valuation function v_i satisfies the *gross substitutes* property if for every $p \preceq p'$ and for every $S \in w_i(p)$, if $S' = \{j \in S : p_j = p'_j\}$, then there exits $S^* \in w_i(p')$ such that $S' \subseteq S^*$.
In other words, "Raising the prices on goods $j \neq i$ doesn't decrease a bidder's demand for good j".

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2. This is what we need: Any good for which bidder i has not been out-bid on has not had its price raised, and so must still be part of a bundle in bidder i's demand set.
3. Hence, we have:

Theorem

In any market in which all buyers satisfy the gross substitutes condition, Walrasian equilibria exist.

Thanks!

See you next class!