Truthful, Pareto Optimal Exchange Without Money

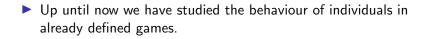
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- Designing the rules of the game to achieve our goals.
- We'll begin our study with the classical "House Allocation Problem" by Shapley and Scarf.
- And study the Top Trading Cycles Algorithm (attributed to David Gale).

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- 4. Doing both is important. If we merely guarantee a "good" allocation, we only know it is "good" w.r.t. reported preferences. But it might be bad w.r.t. real preferences!
- 5. Houses are a toy example. Kidney exchange is a real one (needs a solution without money).

A Model

- 1. There are *n* agents $i \in P$ who each come to market with a good h_i .
- Each agent has a strict preference ordering ≻_i over all of the goods h₁,..., h_n. (i.e. for every pair j, k either h_j ≻_i h_k or h_k ≻_i h_j, and this ordering is transitive so each agent just has a rank order list of goods. In particular, this ranking includes an agents own good h_i.

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We wish to design an algorithm which will induce a game played by the players. The algorithm will take as input the reported preferences \succ_i of each player, and output a permutation μ of the goods. This induces a game: the strategy space for each player is the set of preference orderings \succ_i , the utility function is defined by their true preferences.

What is Good?

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Definition

An allocation μ is *Pareto sub-optimal* if there exists an allocation ν such that for every *i*:

$$\nu(i) \succeq_i \mu(i)$$

and for some *j*;

 $\nu(j) \succ_j \mu(j)$

i.e. everybody is at least as happy with their allocation in ν , and at least one person is strictly happier. In this case, we say that ν Pareto-dominates μ .

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It should not be possible to simultaneously improve for everyone.

What about Incentives?

Definition

A is individually rational if for every player *i*, every preference vector \succ_i , and every set of reports of the other players \succ_{-i} , if $\mu = A(\succ_i, \succ_{-i})$ then:

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Definition

A mechanism A is dominant-strategy incentive compatible if it is a dominant strategy for everyone to report their true preferences. i.e. if for all $\succ_i, \succ_{-i}, \succ'_i$, if

$$\mu = A(\succ_i, \succ_{-i}) \text{ and } \nu = A(\succ'_i, \succ_{-i})$$

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then $\mu(i) \succeq_i \nu(i)$

Top Trading Cycles

Algorithm 1 The top trading cycles algorithm

$$\mathsf{TTC}(\succ_1,\ldots,\succ_n)$$

Let $S_1 = P$ be the set of all agents. Set a counter t = 1. while $|S_i| > 0$ do

Construct a graph $G_t = (V_t, E_t)$ where $V_t = S_t$ and for each $i, j \in V_t$, the directed edge $(i, j) \in E_t$ if and only if $h_j \succ_i h_k$ for all other $k \in V_t$. i.e. this is the graph that results when every agent "points to" their favorite remaining good. **Find** any cycle C_t in G_t and clear all trades along it: i.e. for every directed edge $(i, j) \in C_t$ set $\mu(i) = j$. **Set** $S_{t+1} = S_t$ and remove all cleared agents: for each i: $(i, j) \in C_t$, set $S_{t+1} \leftarrow S_{t+1} - \{i\}$. Increment t $(t \leftarrow t+1)$. **end while**

Output μ .

1. First: establish the algorithm halts at all.

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Lemma

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4. **Proof**: by construction, G_t is a directed graph in which every vertex has out-degree exactly one. (So by starting at any vertex and following edges forward, we must find a cycle).

Interlude: Example

5 agents:

- $\succ_1: 2 \succ 5 \succ 3 \succ 1 \succ 4$
- $\succ_2: 3 \succ 1 \succ 5 \succ 4 \succ 2$
- $\succ_3: 1 \succ 2 \succ 3 \succ 4 \succ 5$
- $\succ_4: 1 \succ 3 \succ 5 \succ 4 \succ 2$
- $\succ_5: 4 \succ 1 \succ 3 \succ 2 \succ 5$

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- $\succ_5: 4 \succ 1 \succ 3 \succ 2 \succ 5$

$$\mu(1) = 2, \mu(2) = 3, \mu(3) = 1, \mu(4) = 5, \mu(5) = 4$$

Theorem

The Top Trading Cycles algorithm produces a Pareto optimal allocation μ on every input \succ .

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- 4. Inductively, if $\nu(i) = \mu(i)$ for every $i \in C_1 \cup \ldots \cup C_k$ for $k \leq t$, then We must also have that $\nu(i) = \mu(i)$ for every $i \in C_{t+1}$.

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5. Continuing through t = n, we have that $\mu = \nu$, a contradiction.

Theorem The Top Trading Cycles algorithm is individually rational.



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Why?



Analysis

Theorem

The Top Trading Cycles Algorithm is Dominant Strategy Incentive Compatible.

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5. But that can't happen...

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3. Agent *i*'s choice set can only increase!

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6. Tada!

Thanks!

See you next class — stay healthy!

