Algorithmic Game Theory: Problem Set 1

Due online via GradeScope before the start of class on Tuesday, February 4

Aaron Roth

Collaboration on problem sets is ok, but list everyone you worked with, and everyone must turn in their own assignment. Ask questions on Slack.

Problem 1) Games with Infinite Action Sets (10 points)

John Nash proved that every game with finitely many players and finitely many actions has a Nash equilibrium in mixed strategies. These conditions are important!

(a) (5 pts) Give an example of a 2 player game in which each player has infinitely many actions and your game has a Nash equilibrium. Precisely describe the equilibrium, and prove that it is a Nash equilibrium.

(b) (5 pts) Give an example of a 2 player game in which each player has infinitely many actions, and prove that your game does not have any Nash equilibrium. *Hint: Don't forget about mixed strategy Nash equilibria!*

Problem 2) Properties of Equilibria (15 pts)

(a) (5 pts) Consider a two-player game G, with a Nash equilibria (p_1, p_2) played by player 1 and player 2, respectively. Note that p_1 and p_2 could each be pure strategies or mixed strategies. Let S_1 be the set of actions in the support of p_1. Prove that $\forall s_i,\ s_j \in S_1$,

$$u_1(s_i, p_2) = u_1(s_j, p_2)$$

In other words, prove that given the opponent's (potentially mixed) strategy p_2, player 1 is indifferent between all pure actions they themselves are randomizing over.

(b) (5 pts) Next, we will prove that while equilibrium implies this indifference condition, this indifference condition does not imply equilibrium. Show a game G and a strategy pair (p_1, p_2) such that $\forall s_i,\ s_j \in S_1$, $u_1(s_i, p_2) = u_1(s_j, p_2)$, and $\forall s_i,\ s_j \in S_2$, $u_2(p_1, s_i) = u_2(p_1, s_j)$, but (p_1, p_2) is *not* an equilibrium.

(c) (5 pts) Recall that a two-person zero-sum game is a game where $u_1(a_i, a_j) = -u_2(a_i, a_j), \forall$ actions pairs a_i, a_j. Consider the same setting as in part a), but assume further that G is zero-sum. Prove that $\forall s_i$, $s_j \in S_2$,

$$u_1(p_1, s_i) = u_1(p_1, s_j)$$

In other words, prove that given their own strategy p_1, player 1 is indifferent between all pure actions player 2 is randomizing over.

Problem 3) Iterated Elimination

Recall that in class we considered one way of solving a game: by iterated elimination of weakly dominated strategies. We can also consider iterated elimination of *strictly* dominated strategies. An action $a_i \in A_i$ is *strictly dominated* if $u_i(a_i, a_{-i}) < u_i(a'_i, a_{-i})$ for some $a'_i \in A$ and for all $a_{-i} \in A_{-i}$. (*i.e.* the inequality is always strict.)

Algorithm 1 Iterated Elimination of Strictly Dominated Strategies

IteratedElim$(A_1, \ldots, A_n, u_1, \ldots, u_n)$.
  Initialize a counter $t = 0$
  For each i, **Let** $B_i^t = A_i$
  while TRUE **do**
    For each i let:

$$\text{Dom}_t^i = \{a_i \in B_i^t \text{ such that there exists } a'_i \in B_i^t \text{ such that for all } s \in B_1^t \times \ldots \times B_n^t,\ u_i(a'_i, s_{-i}) > u_i(a_i, s_{-i})\}$$

    if There exists an i such that $\text{Dom}_t^i \neq \emptyset$ **then**
      Let $B_i^{t+1} = B_i^t - \text{Dom}_t^i$
      Update $t = t + 1$
    else
      Break
    end if
  end while
  Return $B_1^t, \ldots, B_n^t$.

We can write this method as an algorithm, which takes as input a set of n action sets $A_1, \ldots, A_n$ and a set of n utility functions $u_1, \ldots, u_n$, where each u_i is a function $u_i : A_1 \times \ldots \times A_n \to \mathbb{R}$.

Part 1

(9 pts) Consider the following 2 player game.

	A	B	C
X	$1,3$	$2,0$	$0,5$
Y	$3,4$	$2,3$	$1,1$
Z	$2,0$	$4,2$	$1,1$

(a) (2 pts)Which strategies survive iterated elimination of strictly dominated strategies?

(b) (2 pts) What are the pure strategy Nash equilibria of the game?

(c) (5 pts) Find a non-trivial (*i.e.* someone should be randomizing and not just playing a pure strategy) mixed-strategy Nash equilibrium of the game.

Part 2

(20 pts)

(a) (5 pts) Prove that if only a single strategy profile s survives iterated elimination of weakly dominated strategies (*i.e.* if at the end for all i, $|B_i^t| = 1$ and $s_i \in B_i^t$ is the surviving action of player i) then s is a pure strategy Nash equilibrium of the game.

(b) (5 pts) Prove that if only a single strategy profile s survives iterated elimination of *strictly* dominated strategies, then it is the unique pure strategy Nash equilibrium of the game.

(c) (5 pts) Give an example of a game that has two pure strategy Nash equilibria, and depending on the order in which actions are chosen for elimination, *either* of them can be selected as the single surviving strategy profile when we apply iterated elimination of weakly dominated strategies.

(d) (5 pts) Consider the following game, "Guess Two-Thirds the Average", in which each player submits a real number from 0 to 100, and the player whose submission is closest to two-thirds of the average submission wins. Formally, $|P| = n$, and for each player $i \in P$, $A_i = [0, 100]$. Given a collection of actions $a \in A$, let $w(a) = \frac{2}{3n}\sum_{i=1}^{n} a_i$, and let $win(a) = \arg\min_{i \in P} |a_i - w(a)|$ be the set of players whose submissions are closest to 2/3 the average. The utility function for each player is such that $u_i(a) = 1/|win(a)|$ if $i \in win(a)$, and $u_i(a) = 0$ otherwise. Find the unique Nash equilibrium of this game via iterated elimination of dominated strategies.