Chapter 5
Basics of Projective Geometry

Think geometrically, prove algebraically.
—John Tate

5.1 Why Projective Spaces?

For a novice, projective geometry usually appears to be a bit odd, and it is not obvious to motivate why its introduction is inevitable and in fact fruitful. One of the main motivations arises from algebraic geometry.

The main goal of algebraic geometry is to study the properties of geometric objects, such as curves and surfaces, defined implicitly in terms of algebraic equations. For instance, the equation

$$x^2 + y^2 - 1 = 0$$

defines a circle in $\mathbb{R}^2$. More generally, we can consider the curves defined by general equations

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

of degree 2, known as *conics*. It is then natural to ask whether it is possible to classify these curves according to their generic geometric shape. This is indeed possible. Except for so-called singular cases, we get ellipses, parabolas, and hyperbolas. The same question can be asked for surfaces defined by quadratic equations, known as *quadrics*, and again, a classification is possible. However, these classifications are a bit artificial. For example, an ellipse and a hyperbola differ by the fact that a hyperbola has points at infinity, and yet, their geometric properties are identical, provided that points at infinity are handled properly.

Another important problem is the study of intersection of geometric objects (defined algebraically). For example, given two curves C_1 and C_2 of degree m and n, respectively, what is the number of intersection points of C_1 and C_2? (by degree of the curve we mean the total degree of the defining polynomial).

Well, it depends! Even in the case of lines (when $m = n = 1$), there are three possibilities: either the lines coincide, or they are parallel, or there is a single intersection point. In general, we expect mn intersection points, but some of these points may be missing because they are at infinity, because they coincide, or because they are imaginary.

What begins to transpire is that "points at infinity" cause trouble. They cause exceptions that invalidate geometric theorems (for example, consider the more general versions of the theorems of Pappus and Desargues from Section 2.12), and make it difficult to classify geometric objects. Projective geometry is designed to deal with "points at infinity" and regular points in a uniform way, without making a distinction. Points at infinity are now just ordinary points, and many things become simpler. For example, the classification of conics and quadrics becomes simpler, and intersection theory becomes cleaner (although, to be honest, we need to consider complex projective spaces).

Technically, projective geometry can be defined axiomatically, or by buidling upon linear algebra. Historically, the axiomatic approach came first (see Veblen and Young [28, 29], Emil Artin [1], and Coxeter [7, 8, 5, 6]). Although very beautiful and elegant, we believe that it is a harder approach than the linear algebraic approach. In the linear algebraic approach, all notions are considered up to a scalar. For example, a projective point is really a line through the origin. In terms of coordinates, this corresponds to "homogenizing." For example, the homogeneous equation of a conic is

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0.$$

Now, regular points are points of coordinates (x, y, z) with $z \neq 0$, and points at infinity are points of coordinates $(x, y, 0)$ (with x, y, z not all null, and up to a scalar). There is a useful model (interpretation) of plane projective geometry in terms of the central projection in $\mathbb{R}^3$ from the origin onto the plane $z = 1$. Another useful model is the spherical (or the half-spherical) model. In the spherical model, a projective point corresponds to a pair of antipodal points on the sphere.

As affine geometry is the study of properties invariant under affine bijections, projective geometry is the study of properties invariant under bijective projective maps. Roughly speaking, projective maps are linear maps up to a scalar. In analogy with our presentation of affine geometry, we will define projective spaces, projective subspaces, projective frames, and projective maps. The analogy will fade away when we define the projective completion of an affine space, and when we define duality.

One of the virtues of projective geometry is that it yields a very clean presentation of rational curves and rational surfaces. The general idea is that a plane rational curve is the projection of a simpler curve in a larger space, a polynomial curve in $\mathbb{R}^3$, onto the plane $z = 1$, as we now explain.

Polynomial curves are curves defined parametrically in terms of polynomials. More specifically, if $\mathscr{E}$ is an affine space of finite dimension $n \geq 2$ and $(a_0, (e_1, \ldots, e_n))$ is an affine frame for $\mathscr{E}$, a polynomial curve of degree m is a map $F \colon \mathbb{A} \to \mathscr{E}$ such that

$$F(t) = a_0 + F_1(t)e_1 + \cdots + F_n(t)e_n,$$

for all $t \in \mathbb{A}$, where $F_1(t), \ldots, F_n(t)$ are polynomials of degree at most m.

Although many curves can be defined, it is somewhat embarassing that a circle cannot be defined in such a way. In fact, many interesting curves cannot be defined this way, for example, ellipses and hyperbolas. A rather simple way to extend the class of curves defined parametrically is to allow rational functions instead of polynomials. A *parametric rational curve* of degree m is a function $F\colon \mathbb{A} \to \mathscr{E}$ such that

$$F(t) = a_0 + \frac{F_1(t)}{F_{n+1}(t)} e_1 + \cdots + \frac{F_n(t)}{F_{n+1}(t)} e_n,$$

for all $t \in \mathbb{A}$, where $F_1(t), \ldots, F_n(t), F_{n+1}(t)$ are polynomials of degree at most m. For example, a circle in $\mathbb{A}^2$ can be defined by the rational map

$$F(t) = a_0 + \frac{1-t^2}{1+t^2} e_1 + \frac{2t}{1+t^2} e_2.$$

In the above example, the denominator $F_3(t) = 1 + t^2$ never takes the value 0 when t ranges over $\mathbb{A}$, but consider the following curve in $\mathbb{A}^2$:

$$G(t) = a_0 + \frac{t^2}{t} e_1 + \frac{1}{t} e_2.$$

Observe that $G(0)$ is undefined. The curve defined above is a hyperbola, and for t close to 0, the point on the curve goes toward infinity in one of the two asymptotic directions.

A clean way to handle the situation in which the denominator vanishes is to work in a projective space. Intuitively, this means viewing a rational curve in $\mathbb{A}^n$ as some appropriate projection of a polynomial curve in $\mathbb{A}^{n+1}$, back onto $\mathbb{A}^n$.

Given an affine space $\mathscr{E}$, for any hyperplane H in $\mathscr{E}$ and any point a_0 not in H, the *central projection (or conic projection, or perspective projection) of center a_0 onto H*, is the partial map p defined as follows: For every point x not in the hyperplane passing through a_0 and parallel to H, we define $p(x)$ as the intersection of the line defined by a_0 and x with the hyperplane H.

For example, we can view G as a rational curve in $\mathbb{A}^3$ given by

$$G_1(t) = a_0 + t^2 e_1 + e_2 + t e_3.$$

If we project this curve G_1 (in fact, a parabola in $\mathbb{A}^3$) using the central projection (perspective projection) of center a_0 onto the plane of equation $x_3 = 1$, we get the previous hyperbola. For $t = 0$, the point $G_1(0) = a_0 + e_2$ in $\mathbb{A}^3$ is in the plane of equation $x_3 = 0$, and its projection is undefined. We can consider that $G_1(0) = a_0 + e_2$ in $\mathbb{A}^3$ is projected to infinity in the direction of e_2 in the plane $x_3 = 0$. In the setting of projective spaces, this direction corresponds rigorously to a point at infinity.

Let us verify that the central projection used in the previous example has the desired effect. Let us assume that $\mathscr{E}$ has dimension $n+1$ and that $(a_0, (e_1, \ldots, e_{n+1}))$ is an affine frame for $\mathscr{E}$. We want to determine the coordinates of the central projection $p(x)$ of a point $x \in \mathscr{E}$ onto the hyperplane H of equation $x_{n+1} = 1$ (the center of projection being a_0). If

$$x = a_0 + x_1 e_1 + \cdots + x_n e_n + x_{n+1} e_{n+1},$$

assuming that $x_{n+1} \neq 0$; a point on the line passing through a_0 and x has coordinates of the form $(\lambda x_1, \ldots, \lambda x_{n+1})$; and $p(x)$, the central projection of x onto the hyperplane H of equation $x_{n+1} = 1$, is the intersection of the line from a_0 to x and this hyperplane H. Thus we must have $\lambda x_{n+1} = 1$, and the coordinates of $p(x)$ are

$$\left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}, 1 \right).$$

Note that $p(x)$ is undefined when $x_{n+1} = 0$. In projective spaces, we can make sense of such points.

The above calculation confirms that $G(t)$ is a central projection of $G_1(t)$. Similarly, if we define the curve F_1 in $\mathbb{A}^3$ by

$$F_1(t) = a_0 + (1 - t^2)e_1 + 2te_2 + (1 + t^2)e_3,$$

the central projection of the polynomial curve F_1 (again, a parabola in $\mathbb{A}^3$) onto the plane of equation $x_3 = 1$ is the circle F.

What we just sketched is a general method to deal with rational curves. We can use our "hat construction" to embed an affine space $\mathscr{E}$ into a vector space $\widehat{\mathscr{E}}$ having one more dimension, then construct the projective space $\mathbf{P}(\widehat{\mathscr{E}})$. This turns out to be the "projective completion" of the affine space $\mathscr{E}$. Then we can define a rational curve in $\mathbf{P}(\widehat{\mathscr{E}})$, basically as the central projection of a polynomial curve in $\widehat{\mathscr{E}}$ back onto $\mathbf{P}(\widehat{\mathscr{E}})$. The same approach can be used to deal with rational surfaces. Due to the lack of space, such a presentation is omitted from the main text. However, it can be found in the additional material on the web site; see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`.

More generally, the projective completion of an affine space is a very convenient tool to handle "points at infinity" in a clean fashion.

This chapter contains a brief presentation of concepts of projective geometry. The following concepts are presented: projective spaces, projective frames, homogeneous coordinates, projective maps, projective hyperplanes, multiprojective maps, affine patches. The projective completion of an affine space is presented using the "hat construction." The theorems of Pappus and Desargues are proved, using the method in which points are "sent to infinity." We also discuss the cross-ratio and duality. The chapter ends with a very brief explanation of the use of the complexification of a projective space in order to define the notion of angle and orthogonality in a projective setting. We also include a short section on applications of projective geometry, notably to computer vision (camera calibration), efficient communication, and error-correcting codes.

5.2 Projective Spaces

As in the case of affine geometry, our presentation of projective geometry is rather sketchy and biased toward the algorithmic geometry of curves and surfaces. For a systematic treatment of projective geometry, we recommend Berger [3, 4], Samuel [23], Pedoe [21], Coxeter [7, 8, 5, 6], Beutelspacher and Rosenbaum [2], Fresnel [14], Sidler [24], Tisseron [26], Lehmann and Bkouche [20], Vienne [30], and the classical treatise by Veblen and Young [28, 29], which, although slightly old-fashioned, is definitely worth reading. Emil Artin's famous book [1] contains, among other things, an axiomatic presentation of projective geometry, and a wealth of geometric material presented from an algebraic point of view. Other "oldies but goodies" include the beautiful books by Darboux [9] and Klein [19]. For a development of projective geometry addressing the delicate problem of orientation, see Stolfi [25], and for an approach geared towards computer graphics, see Penna and Patterson [22].

First, we define projective spaces, allowing the field K to be arbitrary (which does no harm, and is needed to allow finite and complex projective spaces). Roughly speaking, every projective concept is a linea–algebraic concept "up to a scalar." For spaces, this is made precise as follows

Definition 5.1. Given a vector space E over a field K, the *projective space* $\mathbf{P}(E)$ *induced by* E is the set $(E-\{0\})/\sim$ of equivalence classes of nonzero vectors in E under the equivalence relation $\sim$ defined such that for all $u,v \in E-\{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \text{ for some } \lambda \in K-\{0\}.$$

The *canonical projection* $p\colon (E-\{0\}) \to \mathbf{P}(E)$ is the function associating the equivalence class $[u]_\sim$ modulo $\sim$ to $u \neq 0$. The *dimension* $\dim(\mathbf{P}(E))$ of $\mathbf{P}(E)$ is defined as follows: If E is of infinite dimension, then $\dim(\mathbf{P}(E)) = \dim(E)$, and if E has finite dimension, $\dim(E) = n \geq 1$ then $\dim(\mathbf{P}(E)) = n-1$.

Mathematically, a projective space $\mathbf{P}(E)$ is a set of equivalence classes of vectors in E. The spirit of projective geometry is to view an equivalence class $p(u) = [u]_\sim$ as an "atomic" object, forgetting the internal structure of the equivalence class. For this reason, it is customary to call an equivalence class $a = [u]_\sim$ a *point* (the entire equivalence class $[u]_\sim$ is collapsed into a single object viewed as a point).

Remarks:

(1) If we view E as an affine space, then for any nonnull vector $u \in E$, since

$$[u]_\sim = \{\lambda u \mid \lambda \in K, \lambda \neq 0\},$$

letting

$$Ku = \{\lambda u \mid \lambda \in K\}$$

denote the subspace of dimension 1 spanned by u, the map

$$[u]_\sim \mapsto Ku$$

from $\mathbf{P}(E)$ to the set of one-dimensional subspaces of E is clearly a bijection, and since subspaces of dimension 1 correspond to lines through the origin in E, we can view $\mathbf{P}(E)$ as the set of lines in E passing through the origin. So, the projective space $\mathbf{P}(E)$ can be viewed as the set obtained from E when lines through the origin are treated as points.

However, this is a somewhat deceptive view. Indeed, depending on the structure of the vector space E, a line (through the origin) in E may be a fairly complex object, and treating a line just as a point is really a mental game. For example, E may be the vector space of real homogeneous polynomials $P(x,y,z)$ of degree 2 in three variables x,y,z (plus the null polynomial), and a "line" (through the origin) in E corresponds to an algebraic curve of degree 2. Lots of details need to be filled in, but roughly speaking, the curve defined by P is the "zero locus of P," i.e., the set of points $(x,y,z) \in \mathbf{P}(\mathbb{R}^3)$ (or perhaps in $\mathbf{P}(\mathbb{C}^3)$) for which $P(x,y,z) = 0$. We will come back to this point in Section 5.4 after having introduced homogeneous coordinates.

More generally, E may be a vector space of homogeneous polynomials of degree m in 3 or more variables (plus the null polynomial), and the lines in E correspond to such objects as algebraic curves, algebraic surfaces, and algebraic varieties. The point of view where a complex object such as a curve or a surface is treated as a point in a (projective) space is actually very fruitful and is one of the themes of algebraic geometry (see Fulton [15] or Harris [16]).

(2) When $\dim(E) = 1$, we have $\dim(\mathbf{P}(E)) = 0$. When $E = \{0\}$, we have $\mathbf{P}(E) = \emptyset$. By convention, we give it the dimension -1.

We denote the projective space $\mathbf{P}(K^{n+1})$ by $\mathbb{P}^n_K$. When $K = \mathbb{R}$, we also denote $\mathbb{P}^n_\mathbb{R}$ by $\mathbb{RP}^n$, and when $K = \mathbb{C}$, we denote $\mathbb{P}^n_\mathbb{C}$ by $\mathbb{CP}^n$. The projective space $\mathbb{P}^0_K$ is a (projective) point. The projective space $\mathbb{P}^1_K$ is called a *projective line*. The projective space $\mathbb{P}^2_K$ is called a *projective plane*.

The projective space $\mathbf{P}(E)$ can be visualized in the following way. For simplicity, assume that $E = \mathbb{R}^{n+1}$, and thus $\mathbf{P}(E) = \mathbb{RP}^n$ (the same reasoning applies to $E = K^{n+1}$, where K is any field).

Let H be the affine hyperplane consisting of all points $(x_1,\ldots,x_{n+1})$ such that $x_{n+1} = 1$. Every nonzero vector u in E determines a line D passing through the origin, and this line intersects the hyperplane H in a unique point a, unless D is parallel to H. When D is parallel to H, the line corresponding to the equivalence class of u can be thought of as a point at infinity, often denoted by u_∞. Thus, the projective space $\mathbf{P}(E)$ can be viewed as the set of points in the hyperplane H, together with points at infinity associated with lines in the hyperplane H_∞ of equation $x_{n+1} = 0$. We will come back to this point of view when we consider the projective completion of an affine space. Figure 5.1 illustrates the above representation of the projective space when $E = \mathbb{R}^3$.

We refer to the above model of $\mathbf{P}(E)$ as the *hyperplane model*. In this model some hyperplane H_∞ (through the origin) in $\mathbb{R}^{n+1}$ is singled out, and the points of $\mathbf{P}(E)$ arising from the hyperplane H_∞ are declared to be "points at infinity." The purpose

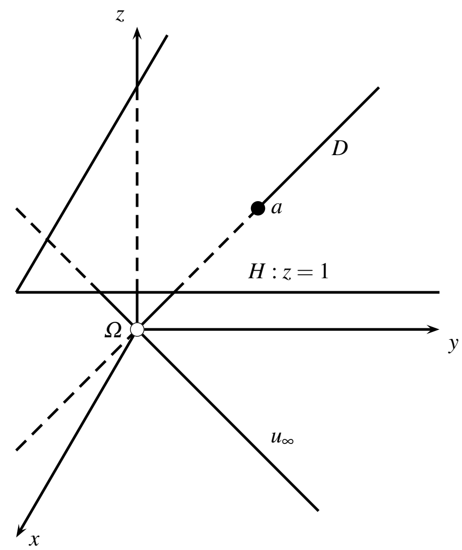


Fig. 5.1 A representation of the projective space $\mathbb{RP}^2$.

of the affine hyperplane H parallel to H_∞ and distinct from H_∞ is to get images for the other points in $\mathbf{P}(E)$ (i.e., those that arise from lines not contained in H_∞). It should be noted that the choice of which points should be considered as infinite is relative to the choice of H_∞. Viewing certain points of $\mathbf{P}(E)$ as points at infinity is convenient for getting a mental picture of $\mathbf{P}(E)$, but there is nothing intrinsic about that. Points of $\mathbf{P}(E)$ are all equal, and unless some additional structure in introduced in $\mathbf{P}(E)$ (such as a hyperplane), a point in $\mathbf{P}(E)$ doesn't know whether it is infinite! The notion of point at infinity is really an affine notion. This point will be made precise in Section 5.6.

Again, for $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$, instead of considering the hyperplane H, we can consider the n-sphere S^n of center 0 and radius 1, i.e., the set of points $(x_1, \ldots, x_{n+1})$ such that

$$x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1.$$

In this case, every line D through the center of the sphere intersects the sphere S^n in two antipodal points a_+ and a_-. The projective space $\mathbb{RP}^n$ is the quotient space obtained from the sphere S^n by identifying antipodal points a_+ and a_-. It is hard to visualize such an object! Nevertheless, some nice projections in $\mathbb{A}^3$ of an embedding of $\mathbb{RP}^2$ into $\mathbb{A}^4$ are given in the surface gallery on the web cite (see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`, Section 24.7). We call this model of $\mathbf{P}(E)$ the *spherical model*.

A more subtle construction consists in considering the (upper) half-sphere instead of the sphere, where the upper half-sphere S^n_+ is set of points on the sphere S^n such that $x_{n+1} \geq 0$. This time, every line through the center intersects the (upper) half-sphere in a single point, except on the boundary of the half-sphere, where it intersects in two antipodal points a_+ and a_-. Thus, the projective space $\mathbb{RP}^n$ is the quotient space obtained from the (upper) half-sphere S^n_+ by identifying antipodal points a_+ and a_- on the boundary of the half-sphere. We call this model of $\mathbf{P}(E)$ the *half-spherical model*.

When $n = 2$, we get a circle. When $n = 3$, the upper half-sphere is homeomorphic to a closed disk (say, by orthogonal projection onto the xy-plane), and $\mathbb{RP}^2$ is in bijection with a closed disk in which antipodal points on its boundary (a unit circle) have been identified. This is hard to visualize! In this model of the real projective space, projective lines are great semicircles on the upper half-sphere, with antipodal points on the boundary identified. Boundary points correspond to points at infinity. By orthogonal projection, these great semicircles correspond to semiellipses, with antipodal points on the boundary identified. Traveling along such a projective "line," when we reach a boundary point, we "wrap around"! In general, the upper half-sphere S^n_+ is homeomorphic to the closed unit ball in $\mathbb{R}^n$, whose boundary is the $(n-1)$-sphere S^{n-1}. For example, the projective space $\mathbb{RP}^3$ is in bijection with the closed unit ball in $\mathbb{R}^3$, with antipodal points on its boundary (the sphere S^2) identified!

Remarks:

(1) A projective space $\mathbf{P}(E)$ has been defined as a *set* without any topological structure. When the field K is either the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers, the vector space E is a topological space. Thus, the projection map $p\colon (E-\{0\}) \to \mathbf{P}(E)$ induces a topology on the projective space $\mathbf{P}(E)$, namely the quotient topology. This means that a subset V of $\mathbf{P}(E)$ is open iff $p^{-1}(V)$ is an open set in E. Then, for example, it turns out that the real projective space $\mathbb{RP}^n$ is homeomorphic to the space obtained by taking the quotient of the (upper) half-sphere S^n_+, by the equivalence relation identifying antipodal points a_+ and a_- on the boundary of the half-sphere. Another interesting fact is that the complex projective line $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$ is homeomorphic to the (real) 2-sphere S^2, and that the real projective space $\mathbb{RP}^3$ is homeomorphic to the group of rotations $\mathbf{SO}(3)$ of $\mathbb{R}^3$.

(2) If H is a hyperplane in E, recall from Lemma 21.1 that there is some nonnull linear form $f \in E^*$ such that $H = \operatorname{Ker} f$. Also, given any nonnull linear form $f \in E^*$, its kernel $H = \operatorname{Ker} f = f^{-1}(0)$ is a hyperplane, and if $\operatorname{Ker} f = \operatorname{Ker} g = H$, then $g = \lambda f$ for some $\lambda \neq 0$. These facts can be concisely stated by saying that the map

$$[f]_\sim \mapsto \operatorname{Ker} f$$

mapping the equivalence class $[f]_\sim = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ to the hyperplane $H = \operatorname{Ker} f$ in E is a bijection between the projective space $\mathbf{P}(E^*)$ and the set of hyperplanes in E. When E is of finite dimension, this bijection yields a useful duality, which will be investigated in Section 5.9.

We now define projective subspaces.

5.3 Projective Subspaces

Projective subspaces of a projective space $\mathbf{P}(E)$ are induced by subspaces of the vector space E.

Definition 5.2. Given a nontrivial vector space E, a *projective subspace (or linear projective variety) of* $\mathbf{P}(E)$ is any subset W of $\mathbf{P}(E)$ such that there is some subspace $V \neq \{0\}$ of E with $W = p(V - \{0\})$. The dimension $\dim(W)$ of W is defined as follows: If V is of infinite dimension, then $\dim(W) = \dim(V)$, and if $\dim(V) = p \geq 1$, then $\dim(W) = p - 1$. We say that a family $(a_i)_{i \in I}$ of points of $\mathbf{P}(E)$ is *projectively independent* if there is a linearly independent family $(\overrightarrow{u_i})_{i \in I}$ in E such that $a_i = p(u_i)$ for every $i \in I$.

Remark: If we allow the empty subset to be a projective subspace, then we have a bijection between the subspaces of E and the projective subspaces of $\mathbf{P}(E)$. In fact, $\mathbf{P}(V)$ is the projective space induced by the vector space V, and we also denote $p(V - \{0\})$ by $\mathbf{P}(V)$, or even by $p(V)$, even though $p(0)$ is undefined.

A projective subspace of dimension 0 is a called a *(projective) point*. A projective subspace of dimension 1 is called a *(projective) line*, and a projective subspace of dimension 2 is called a *(projective) plane*. If H is a hyperplane in E, then $\mathbf{P}(H)$ is called a *projective hyperplane*. It is easily verified that any arbitrary intersection of projective subspaces is a projective subspace. A single point is projectively independent. Two points a, b are projectively independent if $a \neq b$. Two distinct points define a (unique) projective line. Three points a, b, c are projectively independent if they are distinct, and neither belongs to the projective line defined by the other two. Three projectively independent points define a (unique) projective plane.

A closer look at projective subspaces will show some of the advantages of projective geometry: In considering intersection properties, there are no exceptions due to parallelism, as in affine spaces.

Let E be a nontrivial vector space. Given any nontrivial subset S of E, the subset S defines a subset $U = p(S - \{0\})$ of the projective space $\mathbf{P}(E)$, and if $\langle S \rangle$ denotes the subspace of E spanned by S, it is immediately verified that $\mathbf{P}(\langle S \rangle)$ is the intersection of all projective subspaces containing U, and this projective subspace is denoted by $\langle U \rangle$. Given any subspaces M and N of E, recall from Lemma 2.14 that we have the *Grassmann relation*

$$\dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N).$$

Then the following lemma is easily shown.

Lemma 5.1. *Given a projective space* $\mathbf{P}(E)$, *for any two projective subspaces* U, V *of* $\mathbf{P}(E)$, *we have*

$$\dim(U) + \dim(V) = \dim(\langle U \cup V \rangle) + \dim(U \cap V).$$

Furthermore, if $\dim(U) + \dim(V) \geq \dim(\mathbf{P}(E))$*, then* $U \cap V$ *is nonempty and if* $\dim(\mathbf{P}(E)) = n$*, then:*

(i) The intersection of any n hyperplanes is nonempty.
(ii) For every hyperplane H and every point $a \notin H$*, every line D containing a intersects H in a unique point.*
(iii) In a projective plane, every two distinct lines intersect in a unique point.

As a corollary, in the projective space ($\dim(\mathbf{P}(E)) = 3$), for every plane H, every line not contained in H intersects H in a unique point.

It is often useful to deal with projective hyperplanes in terms of nonnull linear forms and equations. Recall that the map

$$[f]_{\sim} \mapsto \mathrm{Ker}\, f$$

is a bijection between $\mathbf{P}(E^*)$ and the set of hyperplanes in E, mapping the equivalence class $[f]_{\sim} = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ to the hyperplane $H = \mathrm{Ker}\, f$. Furthermore, if $u \sim v$, which means that $u = \lambda v$ for some $\lambda \neq 0$, we have

$$f(u) = 0 \quad \text{iff} \quad f(v) = 0,$$

since $f(v) = \lambda f(u)$ and $\lambda \neq 0$. Thus, there is a bijection

$$\{\lambda f \mid \lambda \neq 0\} \mapsto \mathbf{P}(\mathrm{Ker}\, f)$$

mapping points in $\mathbf{P}(E^*)$ to hyperplanes in $\mathbf{P}(E)$. Any nonnull linear form f associated with some hyperplane $\mathbf{P}(H)$ in the above bijection (i.e., $H = \mathrm{Ker}\, f$) is called an *equation of the projective hyperplane* $\mathbf{P}(H)$. We also say that $f = 0$ *is the equation of the hyperplane* $\mathbf{P}(H)$.

Before ending this section, we give an example of a projective space where lines have a nontrivial geometric interpretation, namely as "pencils of lines." If $E = \mathbb{R}^3$, recall that the dual space E^* is the set of all linear maps $f\colon \mathbb{R}^3 \to \mathbb{R}$. As we have just explained, there is a bijection

$$p(f) \mapsto \mathbf{P}(\mathrm{Ker}\, f)$$

between $\mathbf{P}(E^*)$ and the set of lines in $\mathbf{P}(E)$, mapping every point $a = p(f)$ to the line $D_a = \mathbf{P}(\mathrm{Ker}\, f)$.

Is there a way to give a geometric interpretation in $\mathbf{P}(E)$ of a line Δ in $\mathbf{P}(E^*)$? Well, a line Δ in $\mathbf{P}(E^*)$ is defined by two distinct points $a = p(f)$ and $b = p(g)$, where $f, g \in E^*$ are two linearly independent linear forms. But f and g define two distinct planes $H_1 = \mathrm{Ker}\, f$ and $H_2 = \mathrm{Ker}\, g$ through the origin (in $E = \mathbb{R}^3$), and H_1 and H_2 define two distinct lines $D_1 = p(H_1)$ and $D_2 = p(H_2)$ in $\mathbf{P}(E)$. The line Δ in $\mathbf{P}(E^*)$ is of the form $\Delta = p(V)$, where

$$V = \{\lambda f + \mu g \mid \lambda, \mu \in \mathbb{R}\}$$

is the plane in E^* spanned by f, g. Every nonnull linear form $\lambda f + \mu g \in V$ defines a plane $H = \mathrm{Ker}(\lambda f + \mu g)$ in E, and since H_1 and H_2 (in E) are distinct, they intersect in a line L that is also contained in every plane H as above. Thus, the set of planes in E associated with nonnull linear forms in V is just the set of all planes containing the line L. Passing to $\mathbf{P}(E)$ using the projection p, the line L in E corresponds to the point $c = p(L)$ in $\mathbf{P}(E)$, which is just the intersection of the lines D_1 and D_2. Thus, every point of the line Δ in $\mathbf{P}(E^*)$ corresponds to a line in $\mathbf{P}(E)$ passing through c (the intersection of the lines D_1 and D_2), and this correspondence is bijective.

In summary, a line Δ in $\mathbf{P}(E^*)$ corresponds to the set of all lines in $\mathbf{P}(E)$ through some given point. Such sets of lines are called *pencils of lines*.

The above discussion can be generalized to higher dimensions and is discussed quite extensively in Section 5.9. In brief, letting $E = \mathbb{R}^{n+1}$, there is a bijection mapping points in $\mathbf{P}(E^*)$ to hyperplanes in $\mathbf{P}(E)$. A line in $\mathbf{P}(E^*)$ corresponds to a *pencil of hyperplanes* in $\mathbf{P}(E)$, i.e., the set of all hyperplanes containing some given projective subspace $W = p(V)$ of dimension $n-2$. For $n = 3$, a pencil of planes in $\mathbb{RP}^3 = \mathbf{P}(\mathbb{R}^4)$ is the set of all planes (in $\mathbb{RP}^3$) containing some given line W. Other examples of unusual projective spaces and pencils will be given in Section 5.4.

Next, we define the projective analogues of bases (or frames) and linear maps.

5.4 Projective Frames

As all good notions in projective geometry, the concept of a projective frame turns out to be uniquely defined up to a scalar.

Definition 5.3. Given a nontrivial vector space E of dimension $n+1$, a family $(a_i)_{1 \le i \le n+2}$ of $n+2$ points of the projective space $\mathbf{P}(E)$ is a *projective frame (or basis) of* $\mathbf{P}(E)$ if there exists some basis $(e_1, \ldots, e_{n+1})$ of E such that $a_i = p(e_i)$ for $1 \le i \le n+1$, and $a_{n+2} = p(e_1 + \cdots + e_{n+1})$. Any basis with the above property is said to be *associated with the projective frame* $(a_i)_{1 \le i \le n+2}$.

The justification of Definition 5.3 is given by the following lemma.

Lemma 5.2. *If $(a_i)_{1 \le i \le n+2}$ is a projective frame of $\mathbf{P}(E)$, for any two bases $(u_1, \ldots, u_{n+1})$, $(v_1, \ldots, v_{n+1})$ of E such that $a_i = p(u_i) = p(v_i)$ for $1 \le i \le n+1$, and $a_{n+2} = p(u_1 + \cdots + u_{n+1}) = p(v_1 + \cdots + v_{n+1})$, there is a nonzero scalar $\lambda \in K$ such that $v_i = \lambda u_i$, for all i, $1 \le i \le n+1$.*

Proof. Since $p(u_i) = p(v_i)$ for $1 \le i \le n+1$, there exist some nonzero scalars $\lambda_i \in K$ such that $v_i = \lambda_i u_i$ for all i, $1 \le i \le n+1$. Since we must have

$$p(u_1 + \cdots + u_{n+1}) = p(v_1 + \cdots + v_{n+1}),$$

there is some $\lambda \neq 0$ such that

$$\lambda(u_1 + \cdots + u_{n+1}) = v_1 + \cdots + v_{n+1} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

and thus we have

$$(\lambda - \lambda_1)u_1 + \cdots + (\lambda - \lambda_{n+1})u_{n+1} = 0,$$

and since $(u_1, \ldots, u_{n+1})$ is a basis, we have $\lambda_i = \lambda$ for all i, $1 \leq i \leq n+1$, which implies $\lambda_1 = \cdots = \lambda_{n+1} = \lambda$. $\square$

Lemma 5.2 shows that a projective frame determines a unique basis of E, up to a (nonzero) scalar. This would not necessarily be the case if we did not have a point a_{n+2} such that $a_{n+2} = p(u_1 + \cdots + u_{n+1})$.

When $n = 0$, the projective space consists of a single point a, and there is only one projective frame, the pair (a, a). When $n = 1$, the projective space is a line, and a projective frame consists of any three pairwise distinct points a, b, c on this line. When $n = 2$, the projective space is a plane, and a projective frame consists of any four distinct points a, b, c, d such that a, b, c are the vertices of a nondegenerate triangle and d is not on any of the lines determined by the sides of this triangle. The reader can easily generalize to higher dimensions.

Given a projective frame $(a_i)_{1 \leq i \leq n+2}$ of $\mathbf{P}(E)$, let $(u_1, \ldots, u_{n+1})$ be a basis of E associated with $(a_i)_{1 \leq i \leq n+2}$. For every $a \in \mathbf{P}(E)$, there is some $u \in E - \{0\}$ such that

$$a = [u]_\sim = \{\lambda u \mid \lambda \in K - \{0\}\},$$

the equivalence class of u, and the set

$$\{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid v = x_1 u_1 + \cdots + x_{n+1} u_{n+1},\ v \in [u]_\sim = a\}$$

of coordinates of all the vectors in the equivalence class $[u]_\sim$ is called the *set of homogeneous coordinates of a over the basis* $(u_1, \ldots, u_{n+1})$.

Note that for each homogeneous coordinate $(x_1, \ldots, x_{n+1})$ we must have $x_i \neq 0$ for some i, $1 \leq i \leq n+1$, and any two homogeneous coordinates $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ for a differ by a nonzero scalar, i.e., there is some $\lambda \neq 0$ such that $y_i = \lambda x_i$, $1 \leq i \leq n+1$. Homogeneous coordinates $(x_1, \ldots, x_{n+1})$ are sometimes denoted by $(x_1 : \cdots : x_{n+1})$, for instance in algebraic geometry.

By Lemma 5.2, any other basis $(v_1, \ldots, v_{n+1})$ associated with the projective frame $(a_i)_{1 \leq i \leq n+2}$ differs from $(u_1, \ldots, u_{n+1})$ by a nonzero scalar, which implies that the set of homogeneous coordinates of $a \in \mathbf{P}(E)$ over the basis $(v_1, \ldots, v_{n+1})$ is identical to the set of homogeneous coordinates of $a \in \mathbf{P}(E)$ over the basis $(u_1, \ldots, u_{n+1})$. Consequently, we can associate a unique set of homogeneous coordinates to every point $a \in \mathbf{P}(E)$ with respect to the projective frame $(a_i)_{1 \leq i \leq n+2}$. With respect to this projective frame, note that a_{n+2} has homogeneous coordinates $(1, \ldots, 1)$, and that a_i has homogeneous coordinates $(0, \ldots, 1, \ldots, 0)$, where the 1 is in the ith position, where $1 \leq i \leq n+1$. We summarize the above discussion in the following definition.

Definition 5.4. Given a nontrivial vector space E of dimension $n+1$, for any projective frame $(a_i)_{1 \leq i \leq n+2}$ of $\mathbf{P}(E)$ and for any point $a \in \mathbf{P}(E)$, the *set of homogeneous coordinates of a with respect to* $(a_i)_{1 \leq i \leq n+2}$ is the set of $(n+1)$-tuples

$$\{(\lambda x_1, \ldots, \lambda x_{n+1}) \in K^{n+1} \mid x_i \neq 0 \text{ for some } i, \lambda \neq 0,$$
$$a = p(x_1 u_1 + \cdots + x_{n+1} u_{n+1})\},$$

where $(u_1, \ldots, u_{n+1})$ is any basis of E associated with $(a_i)_{1 \leq i \leq n+2}$.

Given a projective frame $(a_i)_{1 \leq i \leq n+2}$ for $\mathbf{P}(E)$, if $(x_1, \ldots, x_{n+1})$ are homogeneous coordinates of a point $a \in \mathbf{P}(E)$, we write $a = (x_1, \ldots, x_{n+1})$, and with a slight abuse of language, we may even talk about a point $(x_1, \ldots, x_{n+1})$ in $\mathbf{P}(E)$ and write $(x_1, \ldots, x_{n+1}) \in \mathbf{P}(E)$.

The special case of the projective line $\mathbb{P}^1_K$ is worth examining. The projective line $\mathbb{P}^1_K$ consists of all equivalence classes $[x, y]$ of pairs $(x, y) \in K^2$ such that $(x, y) \neq (0, 0)$, under the equivalence relation $\sim$ defined such that

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad x_2 = \lambda x_1 \quad \text{and} \quad y_2 = \lambda y_1,$$

for some $\lambda \in K - \{0\}$. When $y \neq 0$, the equivalence class of (x, y) contains the representative $(xy^{-1}, 1)$, and when $y = 0$, the equivalence class of $(x, 0)$ contains the representative $(1, 0)$. Thus, there is a bijection between K and the set of equivalence classes containing some representative of the form $(x, 1)$, and we denote the class $[x, 1]$ by x. The equivalence class $[1, 0]$ is denoted by ∞ and it is called the point at infinity. Thus, the projective line $\mathbb{P}^1_K$ is in bijection with $K \cup \{\infty\}$. The three points $\infty = [1, 0]$, $0 = [0, 1]$, and $1 = [1, 1]$, form a projective frame for $\mathbb{P}^1_K$. The projective frame $(\infty, 0, 1)$ is often called the *canonical frame of* $\mathbb{P}^1_K$.

Homogeneous coordinates are also very useful to handle hyperplanes in terms of equations. If $(a_i)_{1 \leq i \leq n+2}$ is a projective frame for $\mathbf{P}(E)$ associated with a basis $(u_1, \ldots, u_{n+1})$ for E, a nonnull linear form f is determined by $n+1$ scalars $\alpha_1, \ldots, \alpha_{n+1}$ (not all null), and a point $x \in \mathbf{P}(E)$ of homogeneous coordinates $(x_1, \ldots, x_{n+1})$ belongs to the projective hyperplane $\mathbf{P}(H)$ of equation f iff

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0.$$

In particular, if $\mathbf{P}(E)$ is a projective plane, a line is defined by an equation of the form $\alpha x + \beta y + \gamma z = 0$. If $\mathbf{P}(E)$ is a projective space, a plane is defined by an equation of the form $\alpha x + \beta y + \gamma z + \delta w = 0$.

We also have the following lemma giving another characterization of projective frames.

Lemma 5.3. *A family $(a_i)_{1 \leq i \leq n+2}$ of $n+2$ points is a projective frame of $\mathbf{P}(E)$ iff for every i, $1 \leq i \leq n+2$, the subfamily $(a_j)_{j \neq i}$ is projectively independent.*

Proof. We leave as an (easy) exercise the fact that if $(a_i)_{1 \leq i \leq n+2}$ is a projective frame, then each subfamily $(a_j)_{j \neq i}$ is projectively independent. Conversely, pick some $u_i \in E - \{0\}$ such that $a_i = p(u_i)$, $1 \leq i \leq n+2$. Since $(a_j)_{j \neq n+2}$ is projectively independent, $(u_1, \ldots, u_{n+1})$ is a basis of E. Thus, we must have

$$u_{n+2} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

for some $\lambda_i \in K$. However, since for every i, $1 \leq i \leq n+1$, the family $(a_j)_{j \neq i}$ is projectively independent, we must have $\lambda_i \neq 0$, and thus $(\lambda_1 u_1, \ldots, \lambda_{n+1} u_{n+1})$ is also a basis of E, and since

$$u_{n+2} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

it induces the projective frame $(a_i)_{1 \leq i \leq n+2}$. $\square$

Figure 5.2 shows a projective frame (a, b, c, d) in a projective plane. With respect

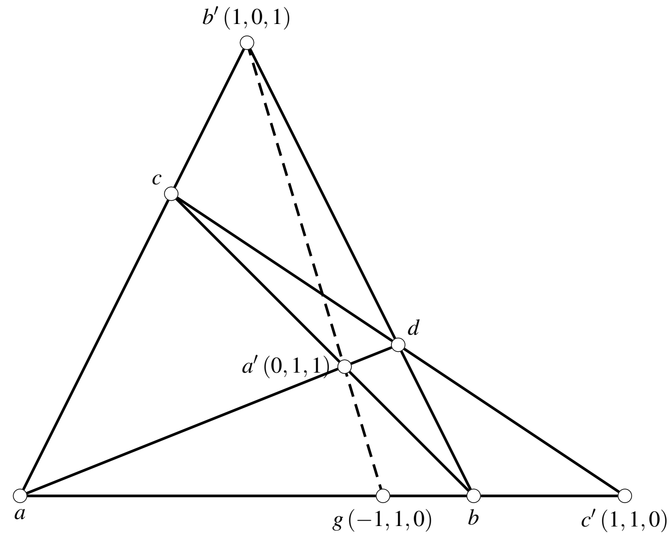


Fig. 5.2 A projective frame (a, b, c, d).

to this projective frame, the points a, b, c, d have homogeneous coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$. Let a' be the intersection of $\langle d, a \rangle$ and $\langle b, c \rangle$, b' be the intersection of $\langle d, b \rangle$ and $\langle a, c \rangle$, and c' be the intersection of $\langle d, c \rangle$ and $\langle a, b \rangle$. Then the points a', b', c' have homogeneous coordinates $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$. The diagram formed by the line segments $\langle a, c' \rangle$, $\langle a, b' \rangle$, $\langle b, b' \rangle$, $\langle c, c' \rangle$, $\langle a, d \rangle$, and $\langle b, c \rangle$ is sometimes called a *Möbius net*. It is easily verified that the equations of the lines $\langle a, b \rangle$, $\langle a, c \rangle$, $\langle b, c \rangle$, are $z = 0$, $y = 0$, and $x = 0$, and the equations of the lines $\langle a, d \rangle$, $\langle b, d \rangle$, and $\langle c, d \rangle$, are $y = z$, $x = z$, and $x = y$. If we let e be the intersection of $\langle b, c \rangle$ and $\langle b', c' \rangle$, f be the intersection of $\langle a, c \rangle$ and $\langle a', c' \rangle$, and g be the intersection of $\langle a, b \rangle$ and $\langle a', b' \rangle$, then it easily seen that e, f, g have homogeneous coordinates $(0,-1,1)$, $(1,0,-1)$, and $(-1,1,0)$. These coordinates satisfy the equation $x + y + z = 0$, which shows that the points e, f, g are collinear. This is a special case of the projective version of Desargues's theorem. This line is called the *polar line (or fundamental line)* of d with respect to the triangle (a, b, c). The diagram also shows the intersection g of $\langle a, b \rangle$ and $\langle a', b' \rangle$.

The projective space of circles provides a nice illustration of homogeneous coordinates. Let E be the vector space (over $\mathbb{R}$) consisting of all homogeneous polynomials of degree 2 in x, y, z of the form

$$ax^2 + ay^2 + bxz + cyz + dz^2$$

(plus the null polynomial). The projective space $\mathbf{P}(E)$ consists of all equivalence classes

$$[P]_\sim = \{\lambda P \mid \lambda \neq 0\},$$

where $P(x,y,z)$ is a nonnull homogeneous polynomial in E. We want to give a geometric interpretation of the points of the projective space $\mathbf{P}(E)$. In order to do so, pick some projective frame (a_1, a_2, a_3, a_4) for the projective plane $\mathbb{RP}^2$, and associate to every $[P] \in \mathbf{P}(E)$ the subset of $\mathbb{RP}^2$ known as its its *zero locus (or zero set, or variety)* $V([P])$, and defined such that

$$V([P]) = \{a \in \mathbb{RP}^2 \mid P(x,y,z) = 0\},$$

where (x,y,z) are homogeneous coordinates for a.

As explained earlier, we also use the simpler notation

$$V([P]) = \{(x,y,z) \in \mathbb{RP}^2 \mid P(x,y,z) = 0\}.$$

Actually, in order for $V([P])$ to make sense, we have to check that $V([P])$ does not depend on the representative chosen in the equivalence class $[P] = \{\lambda P \mid \lambda \neq 0\}$. This is because

$$P(x,y,z) = 0 \quad \text{iff} \quad \lambda P(x,y,z) = 0 \quad \text{when } \lambda \neq 0.$$

For simplicity of notation, we also denote $V([P])$ by $V(P)$. We also have to check that if $(\lambda x, \lambda y, \lambda z)$ are other homogeneous coordinates for $a \in \mathbb{RP}^2$, where $\lambda \neq 0$, then

$$P(x,y,z) = 0 \quad \text{iff} \quad P(\lambda x, \lambda y, \lambda z) = 0.$$

However, since $P(x,y,z)$ is homogeneous of degree 2, we have

$$P(\lambda x, \lambda y, \lambda z) = \lambda^2 P(x,y,z),$$

and since $\lambda \neq 0$,

$$P(x,y,z) = 0 \quad \text{iff} \quad \lambda^2 P(x,y,z) = 0.$$

The above argument applies to any homogeneous polynomial $P(x_1, \ldots, x_n)$ in n variables of any degree m, since

$$P(\lambda x_1, \ldots, \lambda x_n) = \lambda^m P(x_1, \ldots, x_n).$$

Thus, we can associate to every $[P] \in \mathbf{P}(E)$ the curve $V(P)$ in $\mathbb{RP}^2$. One might wonder why we are considering only homogeneous polynomials of degree 2, and not arbitrary polynomials of degree 2? The first reason is that the polynomials in x,y,z of degree 2 do **not** form a vector space. For example, if $P = x^2 + x$ and $Q = -x^2 + y$, the polynomial $P+Q = x+y$ is not of degree 2. We could consider the set of polynomials of degree ≤ 2, which is a vector space, but now the problem is that $V(P)$ is not necessarily well defined!. For example, if $P(x,y,z) = -x^2 + 1$, we have

$$P(1,0,0) = 0 \quad \text{and} \quad P(2,0,0) = -3,$$

and yet $(2,0,0) = 2(1,0,0)$, so that $P(x,y,z)$ takes different values depending on the representative chosen in the equivalence class $[1,0,0]$. Thus, we are led to restrict ourselves to homogeneous polynomials. Actually, this is usually an advantage more than a disadvantage, because homogeneous polynomials tend to be well behaved. For example, by polarization, they yield multilinear maps.

What are the curves $V(P)$? One way to "see" such curves is to go back to the hyperplane model of $\mathbb{RP}^2$ in terms of the plane H of equation $z = 1$ in $\mathbb{R}^3$. Then the trace of $V(P)$ on H is the circle of equation

$$ax^2 + ay^2 + bx + cy + d = 0.$$

Thus, we may think of $\mathbf{P}(E)$ as a projective space of circles. However, there are some problems. For example, $V(P)$ may be empty! This happens, for instance, for $P(x,y,z) = x^2 + y^2 + z^2$, since the equation

$$x^2 + y^2 + z^2 = 0$$

has only the trivial solution $(0,0,0)$, which does not correspond to any point in $\mathbb{RP}^2$. Indeed, only nonnull vectors in $\mathbb{R}^3$ yield points in $\mathbb{RP}^2$. It is also possible that $V(P)$ is reduced to a single point, for instance when $P(x,y,z) = x^2 + y^2$, since the only homogeneous solution of

$$x^2 + y^2 = 0$$

is $(0,0,1)$. Also, note that the map

$$[P] \mapsto V(P)$$

is not injective. For instance, $P = x^2 + y^2$ and $Q = x^2 + 2y^2$ define the same degenerate circle reduced to the point $(0,0,1)$. We also accept as circles the union of two lines, as in the case

$$(bx + cy + dz)z = 0,$$

where $a = 0$, and even a double line, as in the case

$$z^2 = 0,$$

where $a = b = c = 0$.

A clean way to resolve most of these problems is to switch to homogeneous polynomials over the complex field $\mathbb{C}$ and to consider curves in $\mathbb{CP}^2$. This is what is done in algebraic geometry (see Fulton [15] or Harris [16]). If $P(x,y,z)$ is a homogeneous polynomial over $\mathbb{C}$ of degree 2 (plus the null polynomial), it is easy to show that $V(P)$ is always nonempty, and in fact infinite. It can also be shown that $V(P) = V(Q)$ implies that $Q = \lambda P$ for some $\lambda \in \mathbb{C}$, with $\lambda \neq 0$ (see Samuel [23]). Another advantage of switching to the complex field $\mathbb{C}$ is that the theory of intersection is cleaner. Thus, any two circles that do not contain a common line always intersect in four points, some of which might be multiple points (as in the case of tangent circles). This may seem surprising, since in the real plane, two circles intersect in at most two points. Where are the other two points? They turn out to be the points $(1, \mathrm{i}, 0)$ and $(1, -\mathrm{i}, 0)$, as one can immediately verify. We can think of them as complex points at infinity! Not only are they at infinity, but they are not real. No wonder we cannot see them! We will come back to these points, called the *circular points*, in Section 5.11.

Going back to the vector space E over $\mathbb{R}$, it is worth saying that it can be shown that if $V(P) = V(Q)$ contains at least two points (in which case, $V(P)$ is actually infinite), then $Q = \lambda P$ for some $\lambda \in \mathbb{R}$ with $\lambda \neq 0$. Thus, even over $\mathbb{R}$, the mapping

$$[P] \mapsto V(P)$$

is injective whenever $V(P)$ is neither empty nor reduced to a single point. Note that the projective space $\mathbf{P}(E)$ of circles has dimension 3. In fact, it is easy to show that three distinct points that are not collinear determine a unique circle (see Samuel [23]).

In a similar vein, we can define the projective space of conics $\mathbf{P}(E)$ where E is the vector space (over $\mathbb{R}$) consisting of all homogeneous polynomials of degree 2 in x, y, z,

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2$$

(plus the null polynomial). The curves $V(P)$ are indeed conics, perhaps degenerate. To see this, we can use the hyperplane model of $\mathbb{RP}^2$. The trace of $V(P)$ on the plane of equation $z = 1$ is the conic of equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0.$$

Another way to see that $V(P)$ is a conic is to observe that in $\mathbb{R}^3$,

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0$$

defines a cone with vertex $(0, 0, 0)$, and since its section by the plane $z = 1$ is a conic, all of its sections by planes are conics. The mapping

$$[P] \mapsto V(P)$$

is still injective when E is defined over the ground field $\mathbb{C}$, or if $V(P)$ has at least two points when E is defined over $\mathbb{R}$. Note that the projective space $\mathbf{P}(E)$ of conics has dimension 5. In fact, it is easy to show that five distinct points no four of which are not collinear determine a unique conic (see Samuel [23]).

It is also interesting to see what are lines in the space of circles or in the space of conics. In both cases we get pencils (of circles and conics, respectively). For more details, see Samuel [23], Sidler [24], Tisseron [26], Lehmann and Bkouche [20], Pedoe [21], Coxeter [7, 8], and Veblen and Young [28, 29].

We could also investigate algebraic plane curves of any degree m, by letting E be the vector space of homogeneous polynomials of degree m in x, y, z (plus the null polynomial). The zero locus $V(P)$ of P is defined just as before as

$$V(P) = \{(x,y,z) \in \mathbb{RP}^2 \mid P(x,y,z) = 0\}.$$

Observe that when $m = 1$, since homogeneous polynomials of degree 1 are linear forms, we are back to the case where $E = (\mathbb{R}^3)^*$, the dual space of $\mathbb{R}^3$, and $\mathbf{P}(E)$ can be identified with the set of lines in $\mathbb{RP}^2$. But when $m \geq 3$, things are even worse regarding the injectivity of the map $[P] \mapsto V(P)$. For instance, both $P = xy^2$ and $Q = x^2y$ define the same union of two lines. It is necessary to consider *irreducible* curves, i.e., curves that are defined by irreducible polynomials, and to work over the field $\mathbb{C}$ of complex numbers (recall that a polynomial P is irreducible if it cannot be written as the product $P = Q_1Q_2$ of two polynomials Q_1, Q_2 of degree ≥ 1).

We can also investigate algebraic surfaces in $\mathbb{RP}^3$ (or $\mathbb{CP}^3$), by letting E be the vector space of homogeneous polynomials of degree m in four variables x, y, z, t (plus the null polynomial). We can also consider the zero locus of a set of equations

$$\mathscr{E} = \{P_1 = 0, P_2 = 0, \ldots, P_n = 0\},$$

where $P_1, \ldots, P_n$ are homogeneous polynomials of degree m in x, y, z, t, defined as

$$V(\mathscr{E}) = \{(x,y,z,t) \in \mathbb{RP}^3 \mid P_i(x,y,z,t) = 0,\ 1 \leq i \leq n\}.$$

This way, we can also deal with space curves.

Finally, we can consider homogeneous polynomials $P(x_1, \ldots, x_{N+1})$ in $N+1$ variables and of degree m (plus the null polynomial), and study the subsets of $\mathbb{RP}^N$ (or $\mathbb{CP}^N$) defined as the zero locus of a set of equations

$$\mathscr{E} = \{P_1 = 0, P_2 = 0, \ldots, P_n = 0\},$$

where $P_1, \ldots, P_n$ are homogeneous polynomials of degree m in the variables $x_1, \ldots, x_{N+1}$. For example, it turns out that the set of lines in $\mathbb{RP}^3$ forms a surface of degree 2 in $\mathbb{RP}^5$ (the Klein quadric). However, all this would really take us too far into algebraic geometry, and we simply refer the interested reader to Fulton [15] or Harris [16].

We now consider projective maps.

5.5 Projective Maps

Given two nontrivial vector spaces E and F and a linear map $f\colon E \to F$, observe that for every $u, v \in (E - \mathrm{Ker}\, f)$, if $v = \lambda u$ for some $\lambda \in K - \{0\}$, then $f(v) = \lambda f(u)$, and thus f restricted to $(E - \mathrm{Ker}\, f)$ induces a function $\mathbf{P}(f)\colon (\mathbf{P}(E) - \mathbf{P}(\mathrm{Ker}\, f)) \to \mathbf{P}(F)$ defined such that

$$\mathbf{P}(f)([u]_\sim) = [f(u)]_\sim,$$

as in the following commutative diagram:

$$\begin{array}{ccc} E - \mathrm{Ker}\, f & \xrightarrow{\;f\;} & F - \{0\} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{P}(E) - \mathbf{P}(\mathrm{Ker}\, f) & \xrightarrow[\mathbf{P}(f)]{} & \mathbf{P}(F) \end{array}$$

When f is injective, i.e., when $\mathrm{Ker}\, f = \{0\}$, then $\mathbf{P}(f)\colon \mathbf{P}(E) \to \mathbf{P}(F)$ is indeed a well-defined function. The above discussion motivates the following definition.

Definition 5.5. Given two nontrivial vector spaces E and F, any linear map $f\colon E \to F$ induces a partial map $\mathbf{P}(f)\colon \mathbf{P}(E) \to \mathbf{P}(F)$ called a *projective map*, such that if $\mathrm{Ker}\, f = \{u \in E \mid f(u) = 0\}$ is the kernel of f, then $\mathbf{P}(f)\colon (\mathbf{P}(E) - \mathbf{P}(\mathrm{Ker}\, f)) \to \mathbf{P}(F)$ is a total map defined such that

$$\mathbf{P}(f)([u]_\sim) = [f(u)]_\sim,$$

as in the following commutative diagram:

$$\begin{array}{ccc} E - \mathrm{Ker}\, f & \xrightarrow{\;f\;} & F - \{0\} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{P}(E) - \mathbf{P}(\mathrm{Ker}\, f) & \xrightarrow[\mathbf{P}(f)]{} & \mathbf{P}(F) \end{array}$$

If f is injective, i.e., when $\mathrm{Ker}\, f = \{0\}$, then $\mathbf{P}(f)\colon \mathbf{P}(E) \to \mathbf{P}(F)$ is a total function called a *projective transformation*, and when f is bijective, we call $\mathbf{P}(f)$ a *projectivity, or projective isomorphism, or homography*. The set of projectivities $\mathbf{P}(f)\colon \mathbf{P}(E) \to \mathbf{P}(E)$ is a group called the *projective (linear) group*, and is denoted by $\mathbf{PGL}(E)$.

One should realize that if a linear map $f\colon E \to F$ is not injective, then the projective map $\mathbf{P}(f)\colon \mathbf{P}(E) \to \mathbf{P}(F)$ is only a *partial map*, i.e., it is undefined on $\mathbf{P}(\mathrm{Ker}\, f)$. In particular, if $f\colon E \to F$ is the null map (i.e., $\mathrm{Ker}\, f = E$), the domain of $\mathbf{P}(f)$ is empty and $\mathbf{P}(f)$ is the partial function undefined everywhere. We might want to require in Definition 5.5 that f not be the null map to avoid this degenerate case. Projective maps are often defined only when they are induced by bijective linear maps.

We take a closer look at the projectivities of the projective line $\mathbb{P}^1_K$, since they play a role in the "change of parameters" for projective curves. A projectivity $f\colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ is induced by some bijective linear map $g\colon K^2 \to K^2$ given by some invertible matrix

$$M(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$. Since the projective line $\mathbb{P}^1_K$ is isomorphic to $K \cup \{\infty\}$, it is easily verified that f is defined as follows:

$$c \neq 0 \begin{cases} z \mapsto \dfrac{az+b}{cz+d} & \text{if } z \neq -\dfrac{d}{c}, \\ -\dfrac{d}{c} \mapsto \infty, \\ \infty \mapsto \dfrac{a}{c}; \end{cases} \qquad c = 0 \begin{cases} z \mapsto \dfrac{az+b}{d}, \\ \infty \mapsto \infty. \end{cases}$$

If $K = \mathbb{R}$ or $K = \mathbb{C}$, note that a/c is the limit of $(az+b)/(cz+d)$, as z approaches infinity, and the limit of $(az+b)/(cz+d)$ as z approaches $-d/c$ is ∞ (when $c \neq 0$).

Projections between hyperplanes form an important example of projectivities.

Definition 5.6. Given a projective space $\mathbf{P}(E)$, for any two distinct hyperplanes $\mathbf{P}(H)$ and $\mathbf{P}(H')$, for any point $c \in \mathbf{P}(E)$ neither in $\mathbf{P}(H)$ nor in $\mathbf{P}(H')$, the *projection (or perspectivity) of center c between $\mathbf{P}(H)$ and $\mathbf{P}(H')$* is the map $f\colon \mathbf{P}(H) \to \mathbf{P}(H')$ defined such that for every $a \in \mathbf{P}(H)$, the point $f(a)$ is the intersection of the line $\langle c, a\rangle$ through c and a with $\mathbf{P}(H')$.

Let us verify that f is well–defined and a bijective projective transformation. Since the hyperplanes $\mathbf{P}(H)$ and $\mathbf{P}(H')$ are distinct, the hyperplanes H and H' in E are distinct, and since c is neither in $\mathbf{P}(H)$ nor in $\mathbf{P}(H')$, letting $c = p(u)$ for some nonnull vector $u \in E$, then $u \notin H$ and $u \notin H'$, and thus $E = H \oplus Ku = H' \oplus Ku$. If $\pi\colon E \to H'$ is the linear map (projection onto H' parallel to u) defined such that

$$\pi(w + \lambda u) = w,$$

for all $w \in H'$ and all $\lambda \in K$, since $E = H \oplus Ku = H' \oplus Ku$, the restriction $g\colon H \to H'$ of $\pi\colon E \to H'$ to H is a linear bijection between H and H', and clearly $f = \mathbf{P}(g)$, which shows that f is a projectivity.

Remark: Going back to the linear map $\pi\colon E \to H'$ (projection onto H' parallel to u), note that $\mathbf{P}(\pi)\colon \mathbf{P}(E) \to \mathbf{P}(H')$ is also a projective map, but it is not injective, and thus only a partial map. More generally, given a direct sum $E = V \oplus W$, the projection $\pi\colon E \to V$ onto V parallel to W induces a projective map $\mathbf{P}(\pi)\colon \mathbf{P}(E) \to \mathbf{P}(V)$, and given another direct sum $E = U \oplus W$, the restriction of π to U induces a perspectivity f between $\mathbf{P}(U)$ and $\mathbf{P}(V)$. Geometrically, f is defined as follows: Given any point $a \in \mathbf{P}(U)$, if $\langle \mathbf{P}(W), a\rangle$ is the smallest projective subspace containing $\mathbf{P}(W)$ and a, the point $f(a)$ is the intersection of $\langle \mathbf{P}(W), a\rangle$ with $\mathbf{P}(V)$.

Figure 5.3 illustrates a projection f of center c between two projective lines Δ and Δ' (in the real projective plane).

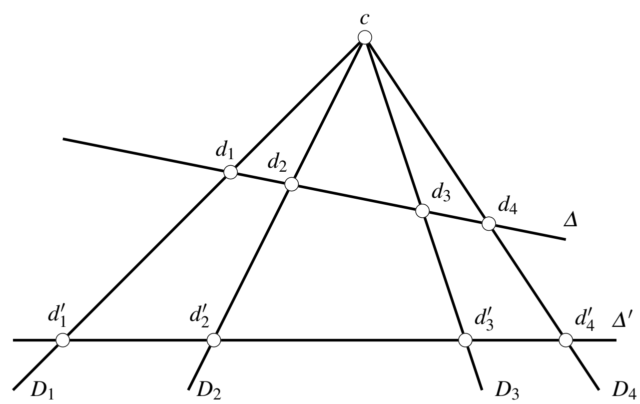


Fig. 5.3 A projection of center c between two lines Δ and Δ'.

If we consider three distinct points d_1, d_2, d_3 on Δ and their images d_1', d_2', d_3' on Δ' under the projection f, then ratios are not preserved, that is,

$$\frac{\overrightarrow{d_3d_1}}{\overrightarrow{d_3d_2}} \neq \frac{\overrightarrow{d_3'd_1'}}{\overrightarrow{d_3'd_2'}}.$$

However, if we consider four distinct points d_1, d_2, d_3, d_4 on Δ and their images d_1', d_2', d_3', d_4' on Δ' under the projection f, we will show later that we have the following preservation of the so-called "cross-ratio"

$$\frac{\overrightarrow{d_3d_1}}{\overrightarrow{d_3d_2}} \Big/ \frac{\overrightarrow{d_4d_1}}{\overrightarrow{d_4d_2}} = \frac{\overrightarrow{d_3'd_1'}}{\overrightarrow{d_3'd_2'}} \Big/ \frac{\overrightarrow{d_4'd_1'}}{\overrightarrow{d_4'd_2'}}.$$

Cross-ratios and projections play an important role in geometry (for some very elegant illustrations of this fact, see Sidler [24]).

We now turn to the issue of determining when two linear maps f, g determine the same projective map, i.e., when $\mathbf{P}(f) = \mathbf{P}(g)$. The following lemma gives us a complete answer.

Lemma 5.4. *Given two nontrivial vector spaces E and F, for any two linear maps $f\colon E \to F$ and $g\colon E \to F$, we have $\mathbf{P}(f) = \mathbf{P}(g)$ iff there is some scalar $\lambda \in K - \{0\}$ such that $g = \lambda f$.*

Proof. If $g = \lambda f$, it is clear that $\mathbf{P}(f) = \mathbf{P}(g)$. Conversely, in order to have $\mathbf{P}(f) = \mathbf{P}(g)$, we must have $\mathrm{Ker}\, f = \mathrm{Ker}\, g$. If $\mathrm{Ker}\, f = \mathrm{Ker}\, g = E$, then f and g are both the null map, and this case is trivial. If $E - \mathrm{Ker}\, f \neq \emptyset$, by taking a basis of $\mathrm{Im}\, f$ and some inverse image of this basis, we obtain a basis B of a subspace G of E such that

$E = \mathrm{Ker}\, f \oplus G$. If $\dim(G) = 1$, the restriction of any linear map $f \colon E \to F$ to G is determined by some nonzero vector $u \in E$ and some scalar $\lambda \in K$, and the lemma is obvious. Thus, assume that $\dim(G) \geq 2$. For any two distinct basis vectors $u, v \in B$, since $\mathbf{P}(f) = \mathbf{P}(g)$, there must be some nonzero scalars $\lambda(u)$, $\lambda(v)$, and $\lambda(u+v)$ such that

$$g(u) = \lambda(u) f(u), \quad g(v) = \lambda(v) f(v), \quad g(u+v) = \lambda(u+v) f(u+v).$$

Since f and g are linear, we get

$$g(u) + g(v) = \lambda(u) f(u) + \lambda(v) f(v) = \lambda(u+v)(f(u) + f(v)),$$

that is,

$$(\lambda(u+v) - \lambda(u)) f(u) + (\lambda(u+v) - \lambda(v)) f(v) = 0.$$

Since f is injective on G and $u, v \in B \subseteq G$ are linearly independent, $f(u)$ and $f(v)$ are also linearly independent, and thus we have

$$\lambda(u+v) = \lambda(u) = \lambda(v).$$

Now we have shown that $\lambda(u) = \lambda(v)$, for any two distinct basis vectors in B, which proves that $\lambda(u)$ is independent of $u \in G$, and proves that $g = \lambda f$. $\square$

Lemma 5.4 shows that the projective linear group $\mathbf{PGL}(E)$ is isomorphic to the quotient group of the linear group $\mathbf{GL}(E)$ modulo the subgroup $K^* \mathrm{id}_E$ (where $K^* = K - \{0\}$). Using projective frames, we prove the following useful result.

Lemma 5.5. *Given two nontrivial vector spaces E and F of the same dimension $n+1$, for any two projective frames $(a_i)_{1 \leq i \leq n+2}$ for $\mathbf{P}(E)$ and $(b_i)_{1 \leq i \leq n+2}$ for $\mathbf{P}(F)$, there is a unique projectivity $h \colon \mathbf{P}(E) \to \mathbf{P}(F)$ such that $h(a_i) = b_i$ for $1 \leq i \leq n+2$.*

Proof. Let $(u_1, \ldots, u_{n+1})$ be a basis of E associated with the projective frame $(a_i)_{1 \leq i \leq n+2}$, and let $(v_1, \ldots, v_{n+1})$ be a basis of F associated with the projective frame $(b_i)_{1 \leq i \leq n+2}$. Since $(u_1, \ldots, u_{n+1})$ is a basis, there is a unique linear bijection $g \colon E \to F$ such that $g(u_i) = v_i$, for $1 \leq i \leq n+1$. Clearly, $h = \mathbf{P}(g)$ is a projectivity such that $h(a_i) = b_i$, for $1 \leq i \leq n+2$. Let $h' \colon \mathbf{P}(E) \to \mathbf{P}(F)$ be any projectivity such that $h'(a_i) = b_i$, for $1 \leq i \leq n+2$. By definition, there is a linear isomorphism $f \colon E \to F$ such that $h' = \mathbf{P}(f)$. Since $h'(a_i) = b_i$, for $1 \leq i \leq n+2$, we must have $f(u_i) = \lambda_i v_i$, for some $\lambda_i \in K - \{0\}$, where $1 \leq i \leq n+1$, and

$$f(u_1 + \cdots + u_{n+1}) = \lambda(v_1 + \cdots + v_{n+1}),$$

for some $\lambda \in K - \{0\}$. By linearity of f, we have

$$\lambda_1 v_1 + \cdots + \lambda_{n+1} v_{n+1} = \lambda v_1 + \cdots + \lambda v_{n+1},$$

and since $(v_1, \ldots, v_{n+1})$ is a basis of F, we must have

$$\lambda_1 = \cdots = \lambda_{n+1} = \lambda.$$

This shows that $f = \lambda g$, and thus that

$$h' = \mathbf{P}(f) = \mathbf{P}(g) = h,$$

and h is uniquely determined. $\square$

The above lemma and Lemma 5.4 are false if K is a skew field. Also, Lemma 5.5 fails if $(b_i)_{1\leq i\leq n+2}$ is not a projective frame, or if a_{n+2} is dropped.

As a corollary of Lemma 5.5, given a projective space $\mathbf{P}(E)$, two distinct projective lines D and D' in $\mathbf{P}(E)$, three distinct points a,b,c on D, and any three distinct points a',b',c' on D', there is a unique projectivity from D to D', mapping a to a', b to b', and c to c'. This is because, as we mentioned earlier, any three distinct points on a line form a projective frame.

Remark: As in the affine case, there is "fundamental theorem of projective geometry." For simplicity, we state this theorem assuming that vector spaces are over the field $K = \mathbb{R}$. Given any two projective spaces $\mathbf{P}(E)$ and $\mathbf{P}(F)$ of the same dimension $n \geq 2$, for any bijective function $f\colon \mathbf{P}(E) \to \mathbf{P}(F)$, if f maps any three distinct collinear points a,b,c to collinear points $f(a), f(b), f(c)$, then f is a projectivity. For more general fields, $f = \mathbf{P}(g)$ for some "semilinear" bijection $g\colon E \to F$. A map such as f (preserving collinearity of any three distinct points) is often called a *collineation*. For $K = \mathbb{R}$, collineations and projectivities coincide. For more details, see Samuel [23].

Before closing this section, we illustrate the power of Lemma 5.5 by proving two interesting results. We begin by characterizing perspectivities between lines.

Lemma 5.6. *Given any two distinct lines D and D' in the real projective plane $\mathbb{RP}^2$, a projectivity $f\colon D \to D'$ is a perspectivity iff $f(O) = O$, where O is the intersection of D and D'.*

Proof. If $f\colon D \to D'$ is a perspectivity, then by the very definition of f, we have $f(O) = O$. Conversely, let $f\colon D \to D'$ be a projectivity such that $f(O) = O$. Let a,b be any two distinct points on D also distinct from O, and let $a' = f(a)$ and $b' = f(b)$ on D'. Since f is a bijection and since a,b,O are pairwise distinct, $a' \neq b'$. Let c be the intersection of the lines $\langle a,a'\rangle$ and $\langle b,b'\rangle$, which by the assumptions on a,b,O, cannot be on D or D'. Then we can define the perspectivity $g\colon D \to D'$ of center c, and by the definition of c, we have

$$g(a) = a', \quad g(b) = b', \quad g(O) = O.$$

However, f agrees with g on O,a,b, and since (O,a,b) is a projective frame for D, by Lemma 5.5, we must have $f = g$. $\square$

Using Lemma 5.6, we can give an elegant proof of a version of Desargues's theorem (in the plane).

Lemma 5.7. *Given two triangles (a,b,c) and (a',b',c') in $\mathbb{RP}^2$, where the points a,b,c,a',b',c' are pairwise distinct and the lines $A = \langle b,c\rangle$, $B = \langle a,c\rangle$, $C = \langle a,b\rangle$, $A' = \langle b',c'\rangle$, $B' = \langle a',c'\rangle$, $C' = \langle a',b'\rangle$ are pairwise distinct, if the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, and $\langle c,c'\rangle$ intersect in a common point d distinct from a,b,c, a',b',c', then the intersection points $p = \langle b,c\rangle \cap \langle b',c'\rangle$, $q = \langle a,c\rangle \cap \langle a',c'\rangle$, and $r = \langle a,b\rangle \cap \langle a',b'\rangle$ belong to a common line distinct from A,B,C, A',B',C'.*

Proof. In view of the assumptions on a,b,c, a',b',c', and d, the point r is on neither $\langle a,a'\rangle$ nor $\langle b,b'\rangle$, the point p is on neither $\langle b,b'\rangle$ nor $\langle c,c'\rangle$, and the point q is on neither $\langle a,a'\rangle$ nor $\langle c,c'\rangle$. It is also immediately shown that the line $\langle p,q\rangle$ is distinct from the lines A,B,C,A',B',C'. Let $f\colon \langle a,a'\rangle \to \langle b,b'\rangle$ be the perspectivity of center r and $g\colon \langle b,b'\rangle \to \langle c,c'\rangle$ be the perspectivity of center p. Let $h = g \circ f$. Since both $f(d) = d$ and $g(d) = d$, we also have $h(d) = d$. Thus by Lemma 5.6, the projectivity $h\colon \langle a,a'\rangle \to \langle c,c'\rangle$ is a perspectivity. Since

$$\begin{aligned} h(a) &= g(f(a)) = g(b) = c,\\ h(a') &= g(f(a')) = g(b') = c', \end{aligned}$$

the intersection q of $\langle a,c\rangle$ and $\langle a',c'\rangle$ is the center of the perspectivity h. Also note that the point $m = \langle a,a'\rangle \cap \langle p,r\rangle$ and its image $h(m)$ are both on the line $\langle p,r\rangle$, since r is the center of f and p is the center of g. Since h is a perspectivity of center q, the line $\langle m,h(m)\rangle = \langle p,r\rangle$ passes through q, which proves the lemma. □

Desargues's theorem is illustrated in Figure 5.4. It can also be shown that every projectivity between two distinct lines is the composition of two perspectivities (not in a unique way). An elegant proof of Pappus's theorem can also be given using perspectivities. For all this and more, the reader is referred to the problems.

We now consider the projective completion of an affine space.

5.6 Projective Completion of an Affine Space, Affine Patches

Given an affine space E with associated vector space $\overrightarrow{E}$, we can form the vector space $\widehat{E}$, the homogenized version of E, and then, the projective space $\mathbf{P}(\widehat{E})$ induced by $\widehat{E}$. This projective space, also denoted by $\widetilde{E}$, has some very interesting properties. In fact, it satisfies a universal property, but before we can say what it is, we have to take a closer look at $\widetilde{E}$.

Since the vector space $\widehat{E}$ is the disjoint union of elements of the form $\langle a,\lambda\rangle$, where $a \in E$ and $\lambda \in K - \{0\}$, and elements of the form $u \in \overrightarrow{E}$, observe that if $\sim$ is the equivalence relation on $\widehat{E}$ used to define the projective space $\mathbf{P}(\widehat{E})$, then the equivalence class $[\langle a,\lambda\rangle]_\sim$ of a weighted point contains the special representative

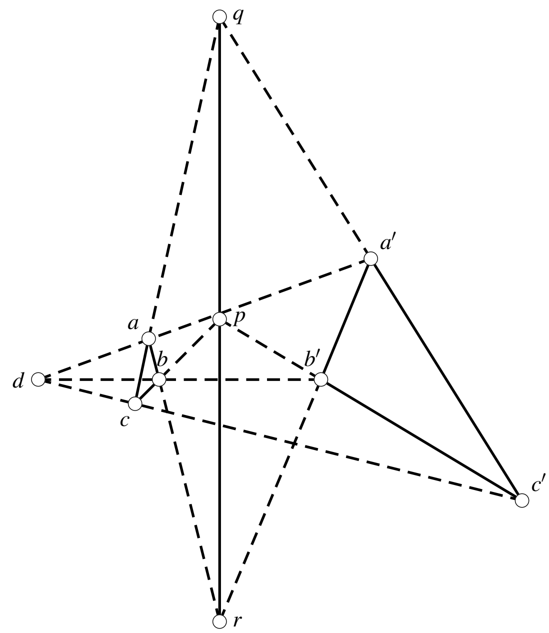


Fig. 5.4 Desargues's theorem (projective version in the plane).

$a = \langle a, 1\rangle$, and the equivalence class $[u]_\sim$ of a nonzero vector $u \in \overrightarrow{E}$ is just a point of the projective space $\mathbf{P}(\overrightarrow{E})$. Thus, there is a bijection

$$\mathbf{P}(\widehat{E}) \longleftrightarrow E \cup \mathbf{P}(\overrightarrow{E})$$

between $\mathbf{P}(\widehat{E})$ and the disjoint union $E \cup \mathbf{P}(\overrightarrow{E})$, which allows us to view E as being embedded in $\mathbf{P}(\widehat{E})$. The points of $\mathbf{P}(\widehat{E})$ in $\mathbf{P}(\overrightarrow{E})$ will be called *points at infinity*, and the projective hyperplane $\mathbf{P}(\overrightarrow{E})$ is called the *hyperplane at infinity*. We will also denote the point $[u]_\sim$ of $\mathbf{P}(\overrightarrow{E})$ (where $u \neq 0$) by u_∞.

Thus, we can think of $\widetilde{E} = \mathbf{P}(\widehat{E})$ as the projective completion of the affine space E obtained by adding points at infinity forming the hyperplane $\mathbf{P}(\overrightarrow{E})$. As we commented in Section 5.2 when we presented the hyperplane model of $\mathbf{P}(E)$, the notion of point at infinity is really an affine notion. But even if a vector space E doesn't arise from the completion of an affine space, there is an affine structure on the complement of any hyperplane $\mathbf{P}(H)$ in the projective space $\mathbf{P}(E)$. In the case of $\widetilde{E}$, the complement E of the projective hyperplane $\mathbf{P}(\overrightarrow{E})$ is indeed an affine space. This is a general property that is needed in order to figure out the universal property of $\widetilde{E}$.

Lemma 5.8. *Given a vector space E and a hyperplane H in E, the complement $E_H = \mathbf{P}(E) - \mathbf{P}(H)$ of the projective hyperplane $\mathbf{P}(H)$ in the projective space $\mathbf{P}(E)$ can be given an affine structure such that the associated vector space of E_H is H. The affine structure on E_H depends only on H, and under this affine structure, E_H is isomorphic to an affine hyperplane in E.*

Proof. Since H is a hyperplane in E, there is some $w \in E - H$ such that $E = Kw \oplus H$. Thus, every vector u in $E - H$ can be written in a unique way as $\lambda w + h$, where $\lambda \neq 0$ and $h \in H$. As a consequence, for every point $[u]$ in E_H, the equivalence class $[u]$ contains a representative of the form $w + \lambda^{-1} h$, with $\lambda \neq 0$. Then we see that the map $\varphi\colon (w + H) \to E_H$, defined such that

$$\varphi(w + h) = [w + h],$$

is a bijection. In order to define an affine structure on E_H, we define $+\colon E_H \times H \to E_H$ as follows: For every point $[w + h_1] \in E_H$ and every $h_2 \in H$, we let

$$[w + h_1] + h_2 = [w + h_1 + h_2].$$

The axioms of an affine space are immediately verified. Now, $w + H$ is an affine hyperplane is E, and under the affine structure just given to E_H, the map $\varphi\colon (w + H) \to E_H$ is an affine map that is bijective. Thus, E_H is isomorphic to the affine hyperplane $w + H$. If we had chosen a different vector $w' \in E - H$ such that $E = Kw' \oplus H$, then E_H would be isomorphic to the affine hyperplane $w' + H$ parallel to $w + H$. But these two hyperplanes are clearly isomorphic by translation, and thus the affine structure on E_H depends only on H. $\square$

An affine space of the form E_H is called an *affine patch* on $\mathbf{P}(E)$. Lemma 5.8 allows us to view a projective space $\mathbf{P}(E)$ as the result of gluing some affine spaces together, at least when E is of finite dimension. For example, when E is of dimension 2, a hyperplane in E is just a line, and the complement of a point in the projective line $\mathbf{P}(E)$ can be viewed as an affine line. Thus, we can view $\mathbf{P}(E)$ as being covered by two affine lines glued together. When $K = \mathbb{R}$, this shows that topologically, the projective line $\mathbb{RP}^1$ is equivalent to a circle. When E is of dimension 3, a hyperplane in E is just a plane, and the complement of a projective line in the projective plane $\mathbf{P}(E)$ can be viewed as an affine plane. Thus, we can view $\mathbf{P}(E)$ as being covered by three affine planes glued together. However, even when $K = \mathbb{R}$, it is much more difficult to come up with a geometric embedding of the projective plane $\mathbb{RP}^2$ in $\mathbb{A}^3$, and in fact, this is impossible! Nevertheless, there are some fascinating immersions of the projective space $\mathbb{RP}^2$ as $3D$ surfaces with self-intersection, one of which is known as the Boy surface. We urge our readers to consult the remarkable book by Hilbert and Cohn-Vossen [17] for drawings of the Boy surface, and more. Some nice projections in $\mathbb{A}^3$ of an embedding of $\mathbb{RP}^2$ into $\mathbb{A}^4$ are given in the surface gallery on the web page (see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`, Section 24.7). In fact, we give a control net in $\mathbb{A}^4$ specifying an explicit rational surface homeomorphic to $\mathbb{RP}^2$. One should also consult Fischer's books [12, 11], where many beautiful models of surfaces are displayed, and the commentaries in

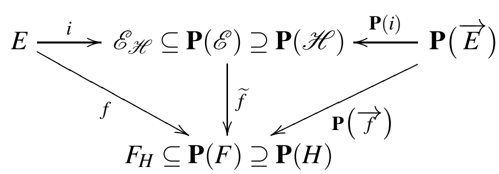
Chapter 6 of [11] regarding models of $\mathbb{RP}^2$. More generally, when E is of dimension $n+1$, the projective space $\mathbf{P}(E)$ is covered by $n+1$ affine patches (hyperplanes) glued together. This idea is very fruitful, since it allows the treatment of projective spaces as manifolds, and it is essential in algebraic geometry.

We can now go back to the projective completion $\widetilde{E}$ of an affine space E.

Definition 5.7. Given any affine space E with associated vector space $\overrightarrow{E}$, a *projective completion of the affine space E with hyperplane at infinity* $\mathbf{P}(\mathscr{H})$ is a triple $\langle \mathbf{P}(\mathscr{E}), \mathbf{P}(\mathscr{H}), i\rangle$, where $\mathscr{E}$ is a vector space, $\mathscr{H}$ is a hyperplane in $\mathscr{E}$, $i\colon E \to \mathbf{P}(\mathscr{E})$ is an injective map such that $i(E) = \mathscr{E}_{\mathscr{H}}$ and i is affine (where $\mathscr{E}_{\mathscr{H}} = \mathbf{P}(\mathscr{E}) - \mathbf{P}(\mathscr{H})$ is an affine patch), and for every projective space $\mathbf{P}(F)$, every hyperplane H in F, and every map $f\colon E \to \mathbf{P}(F)$ such that $f(E) \subseteq F_H$ and f is affine (where $F_H = \mathbf{P}(F) - \mathbf{P}(H)$ is an affine patch), there is a unique projective map $\widetilde{f}\colon \mathbf{P}(\mathscr{E}) \to \mathbf{P}(F)$ such that

$$f = \widetilde{f} \circ i \quad \text{and} \quad \mathbf{P}\big(\overrightarrow{f}\big) = \widetilde{f} \circ \mathbf{P}(i)$$

(where $i\colon \overrightarrow{E} \to \mathscr{H}$ and $\overrightarrow{f}\colon \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i\colon E \to \mathbf{P}(\mathscr{E})$ and $f\colon E \to \mathbf{P}(F)$), as in the following diagram:



The points of $\mathbf{P}(\mathscr{E})$ in $\mathbf{P}(\mathscr{H})$ are called *points at infinity*, and the projective hyperplane $\mathbf{P}(\mathscr{H})$ is called the *hyperplane at infinity*. We will also denote the point $[u]_\sim$ of $\mathbf{P}(\mathscr{H})$ (where $u \neq 0$) by u_∞. As usual, objects defined by a universal property are unique up to isomorphism. We leave the proof as an exercise. The importance of the notion of projective completion stems from the fact that every affine map $f\colon E \to F$ extends *in a unique way* to a projective map $\widetilde{f}\colon \widetilde{E} \to \widetilde{F}$ (provided that the restriction of $\widetilde{f}$ to $\mathbf{P}\big(\overrightarrow{E}\big)$ agrees with $\mathbf{P}\big(\overrightarrow{f}\big)$).

We will now show that $\big\langle \widetilde{E}, \mathbf{P}\big(\overrightarrow{E}\big), i\big\rangle$ is the projective completion of E, where $i\colon E \to \widetilde{E}$ is the injection of E into $\widetilde{E} = E \cup \mathbf{P}\big(\overrightarrow{E}\big)$. For example, if $E = \mathbb{A}^1_K$ is an affine line, its projective completion $\widetilde{\mathbb{A}^1_K}$ is isomorphic to the projective line $\mathbf{P}(K^2)$, and they both can be identified with $\mathbb{A}^1_K \cup \{\infty\}$, the result of adding a point at infinity (∞) to $\mathbb{A}^1_K$. In general, the projective completion $\widetilde{\mathbb{A}^m_K}$ of the affine space $\mathbb{A}^m_K$ is isomorphic to $\mathbf{P}(K^{m+1})$. Thus, $\widetilde{\mathbb{A}^m}$ is isomorphic to $\mathbb{RP}^m$, and $\widetilde{\mathbb{A}^m_{\mathbb{C}}}$ is isomorphic to $\mathbb{CP}^m$.

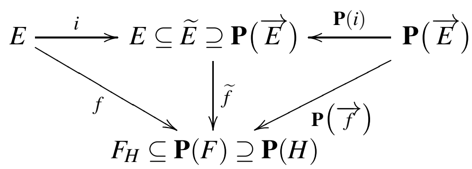
First, let us observe that if E is a vector space and H is a hyperplane in E, then the homogenization $\widehat{E_H}$ of the affine patch E_H (the complement of the projective hyperplane $\mathbf{P}(H)$ in $\mathbf{P}(E)$) is isomorphic to E. The proof is rather simple and uses the fact that there is an affine bijection between E_H and the affine hyperplane $w + H$

in E, where $w \in E - H$ is any fixed vector. Choosing w as an origin in E_H, we know that $\widehat{E_H} = H \widehat{+} Kw$, and since $E = H \oplus Kw$, it is obvious how to define a linear bijection between $\widehat{E_H} = H \widehat{+} Kw$ and $E = H \oplus Kw$. As a consequence the projective spaces $\widetilde{E_H}$ and $\mathbf{P}(E)$ are isomorphic, i.e., there is a projectivity between them.

Lemma 5.9. *Given any affine space $\big(E, \overrightarrow{E}\big)$, for every projective space $\mathbf{P}(F)$, every hyperplane H in F, and every map $f\colon E \to \mathbf{P}(F)$ such that $f(E) \subseteq F_H$ and f is affine (F_H being viewed as an affine patch), there is a unique projective map $\widetilde{f}\colon \widetilde{E} \to \mathbf{P}(F)$ such that*

$$f = \widetilde{f} \circ i \quad \textit{and} \quad \mathbf{P}\big(\overrightarrow{f}\big) = \widetilde{f} \circ \mathbf{P}(i),$$

(where $i\colon \overrightarrow{E} \to \overrightarrow{E}$ and $\overrightarrow{f}\colon \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i\colon E \to \widetilde{E}$ and $f\colon E \to \mathbf{P}(F)$), as in the following diagram:



Proof. The existence of $\widetilde{f}$ is a consequence of Lemma 4.5, where we observe that $\widehat{F_H}$ is isomorphic to F. Just take the projective map $\mathbf{P}\big(\widehat{f}\big)\colon \widetilde{E} \to \mathbf{P}(F)$, where $\widehat{f}\colon \widehat{E} \to F$ is the unique linear map extending f. It remains to prove its uniqueness. Since $f\colon E \to F_H$ is affine, for any $a \in E$ and any $u \in \overrightarrow{E}$, we have

$$f(a+u) = f(a) + \overrightarrow{f}(u),$$

where $\overrightarrow{f}\colon \overrightarrow{E} \to H$ is a linear map. If we fix some $a \in E$, then $f(a) = [w]$, for some $w \in F - H$ and $F = Kw \oplus H$. Assume that $\widetilde{f}\colon \widetilde{E} \to \mathbf{P}(F)$ exists with the desired property. Then there is some linear map $g\colon \widehat{E} \to F$ such that $\widetilde{f} = \mathbf{P}(g)$. Since $f = \widetilde{f} \circ i$, we must have $f(a) = [w] = [g(a)]$, and thus $g(a) = \mu w$, for some $\mu \neq 0$. Also, for every $u \in \overrightarrow{E}$,

$$\begin{aligned} f(a+u) &= [w] + \overrightarrow{f}(u) = \big[w + \overrightarrow{f}(u)\big] = [g(a+u)] \\ &= [g(a) + g(u)] = [\mu w + g(u)], \end{aligned}$$

and thus we must have

$$\lambda(u)w + \lambda(u)\overrightarrow{f}(u) = \mu w + g(u),$$

for some $\lambda(u) \neq 0$. If $\operatorname{Ker} \overrightarrow{f} = \overrightarrow{E}$, the linear map $\overrightarrow{f}$ is the null map, and since we are requiring that the restriction of $\widetilde{f}$ to $\mathbf{P}\big(\overrightarrow{E}\big)$ be equal to $\mathbf{P}\big(\overrightarrow{f}\big)$, the linear map g

must also be the null map on $\overrightarrow{E}$. Thus, $\widetilde{f}$ is unique, and the restriction of $\widetilde{f}$ to $\mathbf{P}(\overrightarrow{E})$ is the partial map undefined everywhere.

If $\overrightarrow{E} - \mathrm{Ker}\,\overrightarrow{f} \neq \emptyset$, by taking a basis of $\mathrm{Im}\,\overrightarrow{f}$ and some inverse image of this basis, we obtain a basis B of a subspace $\overrightarrow{G}$ of $\overrightarrow{E}$ such that $\overrightarrow{E} = \mathrm{Ker}\,\overrightarrow{f} \oplus \overrightarrow{G}$. Since $\overrightarrow{E} = \mathrm{Ker}\,\overrightarrow{f} \oplus \overrightarrow{G}$ where $\dim(\overrightarrow{G}) \geq 1$, for any $x \in \mathrm{Ker}\,\overrightarrow{f}$ and any nonnull vector $y \in \overrightarrow{G}$, we have

$$\begin{aligned} \lambda(x)w &= \mu w + g(x), \\ \lambda(y)w + \lambda(y)\overrightarrow{f}(y) &= \mu w + g(y), \end{aligned}$$

and

$$\lambda(x+y)w + \lambda(x+y)\overrightarrow{f}(x+y) = \mu w + g(x+y),$$

which by linearity yields

$$(\lambda(x+y) - \lambda(x) - \lambda(y) + \mu)w + (\lambda(x+y) - \lambda(y))\overrightarrow{f}(y) = 0.$$

Since $F = Kw \oplus H$ and $\overrightarrow{f}\colon \overrightarrow{E} \to H$, we must have $\lambda(x+y) = \lambda(y)$ and $\lambda(x) = \mu$. Thus, g agrees with $\overrightarrow{f}$ on $\mathrm{Ker}\,\overrightarrow{f}$.

If $\dim(\overrightarrow{G}) = 1$ then for any $y \in \overrightarrow{G}$ we have

$$\lambda(y)w + \lambda(y)\overrightarrow{f}(y) = \mu w + g(y),$$

and for any $\nu \neq 0$ we have

$$\lambda(\nu y)w + \lambda(\nu y)\overrightarrow{f}(\nu y) = \mu w + g(\nu y),$$

which by linearity yields

$$(\lambda(\nu y) - \nu\lambda(y) - \mu + \nu\mu)w + (\nu\lambda(\nu y) - \nu\lambda(y))\overrightarrow{f}(y) = 0.$$

Since $F = Kw \oplus H$, $\overrightarrow{f}\colon \overrightarrow{E} \to H$, and $\nu \neq 0$, we must have $\lambda(\nu y) = \lambda(y)$. Then we must also have $(\lambda(y) - \mu)(1 - \nu) = 0$.

If $K = \{0, 1\}$, since the only nonzero scalar is 1, it is immediate that $g(y) = \overrightarrow{f}(y)$, and we are done. Otherwise, for $\nu \neq 0, 1$, we get $\lambda(y) = \mu$ for all $y \in \overrightarrow{G}$. Then $g = \mu\overrightarrow{f}$ on $\overrightarrow{E}$, and the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}(\overrightarrow{E})$ is equal to $\mathbf{P}(\overrightarrow{f})$. But now g is completely determined by

$$g(u \mathbin{\widehat{+}} \lambda a) = \lambda g(a) + g(u) = \lambda\mu w + \mu\overrightarrow{f}(u).$$

Thus, we have $g = \lambda\widehat{f}$.

Otherwise, if $\dim(\overrightarrow{G}) \geq 2$, then for any two distinct basis vectors u and v in B,

$$\begin{aligned}\lambda(u)w + \lambda(u)\overrightarrow{f}(u) &= \mu w + g(u),\\ \lambda(v)w + \lambda(v)\overrightarrow{f}(v) &= \mu w + g(v),\end{aligned}$$

and

$$\lambda(u+v)w + \lambda(u+v)\overrightarrow{f}(u+v) = \mu w + g(u+v),$$

and by linearity, we get

$$\begin{aligned}(\lambda(u+v) - \lambda(u) - \lambda(v) + \mu)w + (\lambda(u+v) - \lambda(u))\overrightarrow{f}(u)\\ + (\lambda(u+v) - \lambda(v))\overrightarrow{f}(v) = 0.\end{aligned}$$

Since $F = Kw \oplus H$, $\overrightarrow{f}\colon \overrightarrow{E} \to H$, and $\overrightarrow{f}(u)$ and $\overrightarrow{f}(v)$ are linearly independent (because $\overrightarrow{f}$ in injective on $\overrightarrow{G}$), we must have

$$\lambda(u+v) = \lambda(u) = \lambda(v) = \mu,$$

which implies that $g = \mu\overrightarrow{f}$ on $\overrightarrow{E}$, and the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}(\overrightarrow{E})$ is equal to $\mathbf{P}(\overrightarrow{f})$. As in the previous case, g is completely determined by

$$g(u \widehat{+} \lambda a) = \lambda g(a) + g(u) = \lambda\mu w + \mu\overrightarrow{f}(u).$$

Again, we have $g = \lambda\widehat{f}$, and thus $\widetilde{f}$ is unique. $\square$

The requirement that the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}(\overrightarrow{E})$ be equal to $\mathbf{P}(\overrightarrow{f})$ is necessary for the uniqueness of $\widetilde{f}$. The problem comes up when f is a constant map. Indeed, if f is the constant map defined such that $f(a) = [w]$ for some fixed vector $w \in F$, it can be shown that any linear map $g\colon \widehat{E} \to F$ defined such that $g(a) = \mu w$ and $g(u) = \varphi(u)w$ for all $u \in \overrightarrow{E}$, for some $\mu \neq 0$, and some linear form $\varphi\colon \overrightarrow{E} \to F$ satisfies $f = \mathbf{P}(g) \circ i$.

Lemma 5.9 shows that $\langle \widetilde{E}, \mathbf{P}(\overrightarrow{E}), i\rangle$ is the projective completion of the affine space E.

The projective completion $\widetilde{E}$ of an affine space E is a very handy place in which to do geometry in, mainly because the following facts can be easily established.

There is a bijection between affine subspaces of E and projective subspaces of $\widetilde{E}$ not contained in $\mathbf{P}(\overrightarrow{E})$. Two affine subspaces of E are parallel iff the corresponding projective subspaces of $\widetilde{E}$ have the same intersection with the hyperplane at infinity $\mathbf{P}(\overrightarrow{E})$. There is also a bijection between affine maps from E to F and projective maps from $\widetilde{E}$ to $\widetilde{F}$ mapping the hyperplane at infinity $\mathbf{P}(\overrightarrow{E})$ into the hyperplane at infinity $\mathbf{P}(\overrightarrow{F})$. In the projective plane, two distinct lines intersect in a single point (possibly at infinity, when the lines are parallel). In the projective space, two distinct planes intersect in a single line (possibly at infinity, when the planes are parallel). In the projective space, a plane and a line not contained in that plane intersect in a single point (possibly at infinity, when the plane and the line are parallel).

5.7 Making Good Use of Hyperplanes at Infinity

Given a vector space E and a hyperplane H in E, we have already observed that the projective spaces $\widetilde{E_H}$ and $\mathbf{P}(E)$ are isomorphic. Thus, $\mathbf{P}(H)$ can be viewed as the hyperplane at infinity in $\mathbf{P}(E)$, and the considerations applying to the projective completion of an affine space apply to the affine patch E_H on $\mathbf{P}(E)$. This fact yields a powerful and elegant method for proving theorems in projective geometry. The general schema is to choose some projective hyperplane $\mathbf{P}(H)$ in $\mathbf{P}(E)$, view it as the "hyperplane at infinity," then prove an affine version of the desired result in the affine patch E_H (the complement of $\mathbf{P}(H)$ in $\mathbf{P}(E)$, which has an affine structure), and then transfer this result back to the projective space $\mathbf{P}(E)$. This technique is often called "sending objects to infinity." We refer the reader to geometry textbooks for a comprehensive development of these ideas (for example, Berger [3, 4], Samuel [23], Sidler [24], Tisseron [26], or Pedoe [21]), but we cannot resist presenting the projective versions of the theorems of Pappus and Desargues. Indeed, the method of sending points to infinity provides some strikingly elegant proofs. We begin with Pappus's theorem, illustrated in Figure 5.5.

Lemma 5.10. *Given any projective plane $\mathbf{P}(E)$ and any two distinct lines D and D', for any distinct points a,b,c,a',b',c', with a,b,c on D and a',b',c' on D', if a,b,c,a',b',c' are distinct from the intersection of D and D', then the intersection points $p = \langle b,c'\rangle \cap \langle b',c\rangle$, $q = \langle a,c'\rangle \cap \langle a',c\rangle$, and $r = \langle a,b'\rangle \cap \langle a',b\rangle$ are collinear.*

Proof. First, since any two lines in a projective plane intersect in a single point, the points p,q,r are well defined. Choose $\Delta = \langle p,r\rangle$ as the line at infinity, and consider the affine plane $X = \mathbf{P}(E) - \Delta$. Since $\langle a,b'\rangle$ and $\langle a',b\rangle$ intersect at a point at infinity r on Δ, $\langle a,b'\rangle$ and $\langle a',b\rangle$ are parallel, and similarly $\langle b,c'\rangle$ and $\langle b',c\rangle$ are parallel. Thus, by the affine version of Pappus's theorem (Lemma 2.11), the lines $\langle a,c'\rangle$ and $\langle a',c\rangle$ are parallel, which means that their intersection q is on the line at infinity $\Delta = \langle p,r\rangle$, which means that p,q,r are collinear. $\square$

By working in the projective completion of an affine plane, we can obtain an improved version of Pappus's theorem for affine planes. The reader will have to figure out how to deal with the special cases where some of p,q,r go to infinity.

Now, we prove a projective version of Desargues's theorem slightly more general than that given in Lemma 5.7. It is interesting that the proof is radically different, depending on the dimension of the projective space $\mathbf{P}(E)$. This is not surprising. In axiomatic presentations of projective plane geometry, Desargues's theorem is independent of the other axioms. Desargues's theorem is illustrated in Figure 5.6.

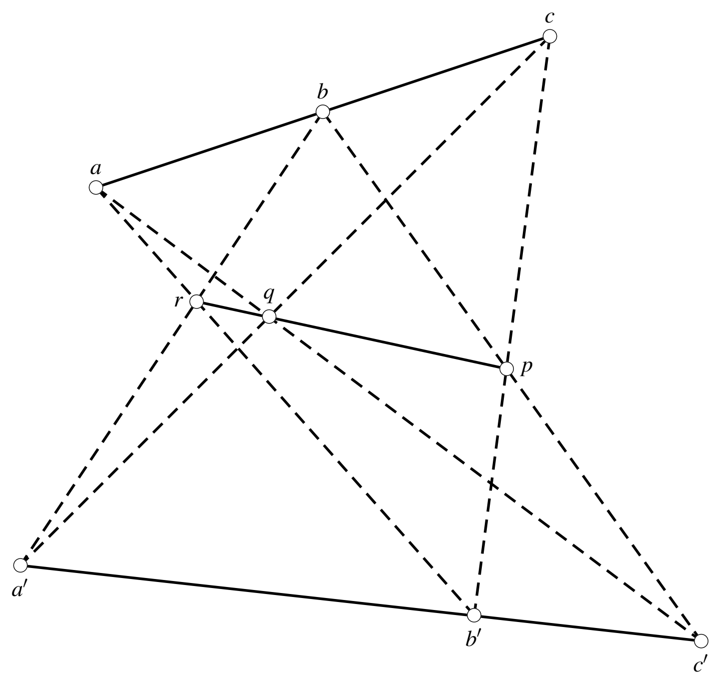


Fig. 5.5 Pappus's theorem (projective version).

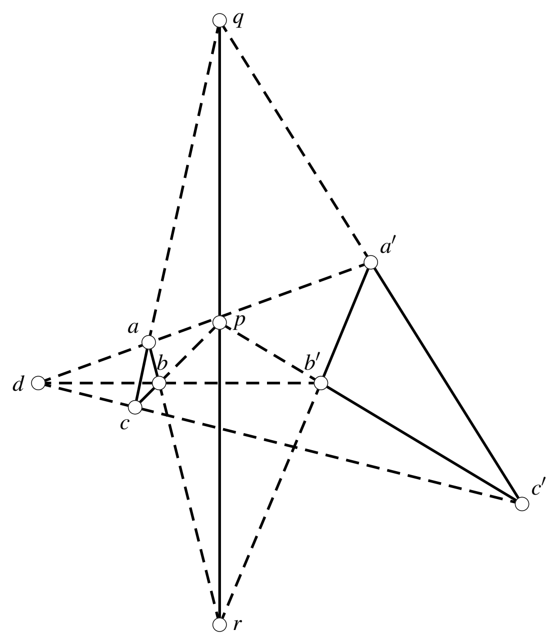


Fig. 5.6 Desargues's theorem (projective version).

Lemma 5.11. *Let $\mathbf{P}(E)$ be a projective space. Given two triangles (a,b,c) and (a',b',c'), where the points a,b,c,a',b',c' are pairwise distinct and the lines $A = \langle b,c\rangle$, $B = \langle a,c\rangle$, $C = \langle a,b\rangle$, $A' = \langle b',c'\rangle$, $B' = \langle a',c'\rangle$, $C' = \langle a',b'\rangle$ are pairwise distinct, if the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, and $\langle c,c'\rangle$ intersect in a common point d distinct from a,b,c, a',b',c', then the intersection points $p = \langle b,c\rangle \cap \langle b',c'\rangle$, $q = \langle a,c\rangle \cap \langle a',c'\rangle$, and $r = \langle a,b\rangle \cap \langle a',b'\rangle$ belong to a common line distinct from A,B,C, A',B',C'.*

Proof. First, it is immediately shown that the line $\langle p,q\rangle$ is distinct from the lines A,B,C, A',B',C'. Let us assume that $\mathbf{P}(E)$ has dimension $n \geq 3$. If the seven points d,a,b,c,a',b',c' generate a projective subspace of dimension 3, then by Lemma 5.1, the intersection of the two planes $\langle a,b,c\rangle$ and $\langle a',b',c'\rangle$ is a line, and thus p,q,r are collinear.

If $\mathbf{P}(E)$ has dimension $n = 2$ or the seven points d,a,b,c,a',b',c' generate a projective subspace of dimension 2, we use the following argument. In the projective plane X generated by the seven points d,a,b,c,a',b',c', choose the projective line $\Delta = \langle p,r\rangle$ as the line at infinity. Then in the affine plane $Y = X - \Delta$, the lines $\langle b,c\rangle$ and $\langle b',c'\rangle$ are parallel, and the lines $\langle a,b\rangle$ and $\langle a',b'\rangle$ are parallel, and the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, and $\langle c,c'\rangle$ are either parallel or concurrent. Then by the converse of the affine version of Desargues's theorem (Lemma 2.12), the lines $\langle a,c\rangle$ and $\langle a',c'\rangle$ are parallel, which means that their intersection q belongs to the line at infinity $\Delta = \langle p,r\rangle$, and thus that p,q,r are collinear. $\square$

The converse of Desargues's theorem also holds (see the problems). Using the projective completion of an affine space, it is easy to state an improved affine version of Desargues's theorem. The reader will have to figure out how to deal with the case where some of the points p,q,r go to infinity. It can also be shown that Pappus's theorem implies Desargues's theorem. Many results of projective or affine geometry can be obtained using the method of "sending points to infinity."

We now discuss briefly the notion of cross-ratio, since it is a major concept of projective geometry.

5.8 The Cross-Ratio

Recall that affine maps preserve the ratio of three collinear points. In general, projective maps do not preserve the ratio of three collinear points. However, bijective projective maps preserve the "ratio of ratios" of any four collinear points (three of which are distinct). Such ratios are called *cross-ratios* (in French, "birapport"). There are several ways of introducing cross-ratios, but since we already have Lemma 5.5 at our disposal, we can circumvent some of the tedious calculations needed if other approaches are chosen.

Given a field K, say $K = \mathbb{R}$, recall that the projective line $\mathbb{P}^1_K$ consists of all equivalence classes $[x,y]$ of pairs $(x,y) \in K^2$ such that $(x,y) \neq (0,0)$, under the equivalence relation $\sim$ defined such that

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad x_2 = \lambda x_1 \quad \text{and} \quad y_2 = \lambda y_1,$$

for some $\lambda \in K - \{0\}$. Letting $\infty = [1,0]$, the projective line $\mathbb{P}^1_K$ is in bijection with $K \cup \{\infty\}$. Furthermore, letting $0 = [0,1]$ and $1 = [1,1]$, the triple $(\infty, 0, 1)$ forms a projective frame for $\mathbb{P}^1_K$. Using this projective frame and Lemma 5.5, we define the cross-ratio of four collinear points as follows.

Definition 5.8. Given a projective line $\Delta = \mathbf{P}(D)$ over a field K, for any sequence (a,b,c,d) of four points in Δ, where a,b,c are distinct (i.e., (a,b,c) is a projective frame), the *cross-ratio* $[a,b,c,d]$ is defined as the element $h(d) \in \mathbb{P}^1_K$, where $h\colon \Delta \to \mathbb{P}^1_K$ is the unique projectivity such that $h(a) = \infty$, $h(b) = 0$, and $h(c) = 1$ (which exists by Lemma 5.5, since (a,b,c) is a projective frame for Δ and $(\infty, 0, 1)$ is a projective frame for $\mathbb{P}^1_K$). For any projective space $\mathbf{P}(E)$ (of dimension ≥ 2) over a field K and any sequence (a,b,c,d) of four collinear points in $\mathbf{P}(E)$, where a,b,c are distinct, the cross-ratio $[a,b,c,d]$ is defined using the projective line Δ that the points a,b,c,d define. For any affine space E and any sequence (a,b,c,d) of four collinear points in E, where a,b,c are distinct, the cross-ratio $[a,b,c,d]$ is defined by considering E as embedded in $\widetilde{E}$.

It should be noted that the definition of the cross-ratio $[a,b,c,d]$ depends on the order of the points. Thus, there could be $24 = 4!$ different possible values depending on the permutation of $\{a,b,c,d\}$. In fact, there are at most 6 distinct values. Also, note that $[a,b,c,d] = \infty$ iff $d = a$, $[a,b,c,d] = 0$ iff $d = b$, and $[a,b,c,d] = 1$ iff $d = c$. Thus, $[a,b,c,d] \in K - \{0,1\}$ iff $d \notin \{a,b,c\}$.

The following lemma is almost obvious, but very important. It shows that projectivities between projective lines are characterized by the preservation of the cross-ratio of any four points (three of which are distinct).

Lemma 5.12. *Given any two projective lines Δ and Δ', for any sequence (a,b,c,d) of points in Δ and any sequence (a',b',c',d') of points in Δ', if a,b,c are distinct and a',b',c' are distinct, there is a unique projectivity $f\colon \Delta \to \Delta'$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(d) = d'$ iff $[a,b,c,d] = [a',b',c',d']$.*

Proof. First, assume that $f\colon \Delta \to \Delta'$ is a projectivity such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(d) = d'$. Let $h\colon \Delta \to \mathbb{P}^1_K$ be the unique projectivity such that $h(a) = \infty$, $h(b) = 0$, and $h(c) = 1$, and let $h'\colon \Delta' \to \mathbb{P}^1_K$ be the unique projectivity such that $h'(a') = \infty$, $h'(b') = 0$, and $h'(c') = 1$. By definition, $[a,b,c,d] = h(d)$ and $[a',b',c',d'] = h'(d')$. However, $h' \circ f\colon \Delta \to \mathbb{P}^1_K$ is a projectivity such that $(h' \circ f)(a) = \infty$, $(h' \circ f)(b) = 0$, and $(h' \circ f)(c) = 1$, and by the uniqueness of h, we get $h = h' \circ f$. But then, $[a,b,c,d] = h(d) = h'(f(d)) = h'(d') = [a',b',c',d']$.

Conversely, assume that $[a,b,c,d] = [a',b',c',d']$. Since (a,b,c) and $(a',\, b',\, c')$ are projective frames, by Lemma 5.5, there is a unique projectivity $g\colon \Delta \to \Delta'$ such that $g(a) = a'$, $g(b) = b'$, and $g(c) = c'$. Now, $h' \circ g\colon \Delta \to \mathbb{P}^1_K$ is a projectivity such that $(h' \circ g)(a) = \infty$, $(h' \circ g)(b) = 0$, and $(h' \circ g)(c) = 1$, and thus, $h = h' \circ g$. However, $h'(d') = [a',b',c',d'] = [a,b,c,d] = h(d) = h'(g(d))$, and since h' is injective, we get $d' = g(d)$. $\square$

As a corollary of Lemma 5.12, given any three distinct points a,b,c on a projective line Δ, for every $\lambda \in \mathbb{P}^1_K$ there is a unique point $d \in \Delta$ such that $[a,b,c,d] = \lambda$.

In order to compute explicitly the cross-ratio, we show the following easy lemma.

Lemma 5.13. *Given any projective line $\Delta = \mathbf{P}(D)$, for any three distinct points a,b,c in Δ, if $a = p(u)$, $b = p(v)$, and $c = p(u+v)$, where (u,v) is a basis of D, and for any $[\lambda,\mu]_\sim \in \mathbb{P}^1_K$ and any point $d \in \Delta$, we have*

$$d = p(\lambda u + \mu v) \quad \textit{iff} \quad [a,b,c,d] = [\lambda,\mu]_\sim.$$

Proof. If (e_1,e_2) is the basis of K^2 such that $e_1 = (1,0)$ and $e_2 = (0,1)$, it is obvious that $p(e_1) = \infty$, $p(e_2) = 0$, and $p(e_1+e_2) = 1$. Let $f\colon D \to K^2$ be the bijective linear map such that $f(u) = e_1$ and $f(v) = e_2$. Then $f(u+v) = e_1+e_2$, and thus f induces the unique projectivity $\mathbf{P}(f)\colon \mathbf{P}(D) \to \mathbb{P}^1_K$ such that $\mathbf{P}(f)(a) = \infty$, $\mathbf{P}(f)(b) = 0$, and $\mathbf{P}(f)(c) = 1$. Then

$$\mathbf{P}(f)(p(\lambda u+\mu v)) = [f(\lambda u+\mu v)]_\sim = [\lambda e_1+\mu e_2]_\sim = [\lambda,\mu]_\sim,$$

that is,

$$d = p(\lambda u + \mu v) \quad \text{iff} \quad [a,b,c,d] = [\lambda,\mu]_\sim.$$

□

We can now compute the cross-ratio explicitly for any given basis (u,v) of D. Assume that a,b,c,d have homogeneous coordinates $[\lambda_1,\mu_1]$, $[\lambda_2,\mu_2]$, $[\lambda_3,\mu_3]$, and $[\lambda_4,\mu_4]$ over the projective frame induced by (u,v). Letting $w_i = \lambda_i u + \mu_i v$, we have $a = p(w_1)$, $b = p(w_2)$, $c = p(w_3)$, and $d = p(w_4)$. Since a and b are distinct, w_1 and w_2 are linearly independent, and we can write $w_3 = \alpha w_1 + \beta w_2$ and $w_4 = \gamma w_1 + \delta w_2$, which can also be written as

$$w_4 = \frac{\gamma}{\alpha}\,\alpha\, w_1 + \frac{\delta}{\beta}\,\beta\, w_2,$$

and by Lemma 5.13, $[a,b,c,d] = \left[\gamma/\alpha, \delta/\beta\right]$. However, since w_1 and w_2 are linearly independent, it is possible to solve for $\alpha,\beta,\gamma,\delta$ in terms of the homogeneous coordinates, obtaining expressions involving determinants:

$$\alpha = \frac{\det(w_3,w_2)}{\det(w_1,w_2)}, \qquad \beta = \frac{\det(w_1,w_3)}{\det(w_1,w_2)},$$
$$\gamma = \frac{\det(w_4,w_2)}{\det(w_1,w_2)}, \qquad \delta = \frac{\det(w_1,w_4)}{\det(w_1,w_2)},$$

and thus, assuming that $d \neq a$, we get

$$[a,b,c,d] = \frac{\begin{vmatrix}\lambda_3 & \lambda_1\\ \mu_3 & \mu_1\end{vmatrix}}{\begin{vmatrix}\lambda_3 & \lambda_2\\ \mu_3 & \mu_2\end{vmatrix}} \Bigg/ \frac{\begin{vmatrix}\lambda_4 & \lambda_1\\ \mu_4 & \mu_1\end{vmatrix}}{\begin{vmatrix}\lambda_4 & \lambda_2\\ \mu_4 & \mu_2\end{vmatrix}}.$$

When $d = a$, we have $[a,b,c,d] = \infty$. In particular, if Δ is the projective completion of an affine line D, then $\mu_i = 1$, and we get

$$[a,b,c,d] = \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2} \bigg/ \frac{\lambda_4 - \lambda_1}{\lambda_4 - \lambda_2} = \frac{\overrightarrow{ca}}{\overrightarrow{cb}} \bigg/ \frac{\overrightarrow{da}}{\overrightarrow{db}}.$$

When $d = \infty$, we get

$$[a,b,c,\infty] = \frac{\overrightarrow{ca}}{\overrightarrow{cb}},$$

which is just the usual ratio (although we defined it as $-\mathrm{ratio}(a,c,b)$).

We briefly mention some of the properties of the cross-ratio. For example, the cross-ratio $[a,b,c,d]$ is invariant if any two elements and the complementary two elements are transposed, and letting $0^{-1} = \infty$ and $\infty^{-1} = 0$, we have

$$[a,b,c,d] = [b,a,c,d]^{-1} = [a,b,d,c]^{-1}$$

and

$$[a,b,c,d] = 1 - [a,c,b,d].$$

Since the permutations of $\{a,b,c,d\}$ are generated by the above transpositions, the cross-ratio takes at most six values. Letting $\lambda = [a,b,c,d]$, if $\lambda \in \{\infty, 0, 1\}$, then any permutation of $\{a,b,c,d\}$ yields a cross-ratio in $\{\infty, 0, 1\}$, and if $\lambda \notin \{\infty, 0, 1\}$, then there are at most the six values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad 1 - \frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda - 1}.$$

We also define when four points form a harmonic division. For this, we need to assume that K is not of characteristic 2.

Definition 5.9. Given a projective line Δ, we say that a sequence of four collinear points (a,b,c,d) in Δ (where a,b,c are distinct) forms a *harmonic division* if $[a,b,c,d] = -1$. When $[a,b,c,d] = -1$, we also say that c and d are *harmonic conjugates* of a and b.

If a,b,c are distinct collinear points in some affine space, from

$$[a,b,c,\infty] = \frac{\overrightarrow{ca}}{\overrightarrow{cb}},$$

we note that c is the midpoint of (a,b) iff $[a,b,c,\infty] = -1$, that is, if (a,b,c,∞) forms a harmonic division. Figure 5.7 shows a harmonic division (a,b,c,d) on the real line, where the coordinates of (a,b,c,d) are $(-2,2,1,4)$.

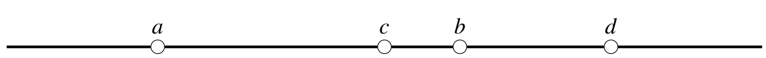


Fig. 5.7 Four points forming a harmonic division.

There is a nice geometric interpretation of harmonic divisions in terms of quadrangles (or complete quadrilaterals). Consider the quadrangle (projective frame) (a,b,c,d) in a projective plane, and let a' be the intersection of $\langle d,a\rangle$ and $\langle b,c\rangle$, b' be the intersection of $\langle d,b\rangle$ and $\langle a,c\rangle$, and c' be the intersection of $\langle d,c\rangle$ and $\langle a,b\rangle$. If we let g be the intersection of $\langle a,b\rangle$ and $\langle a',b'\rangle$, then it is an interesting exercise to show that (a,b,g,c') is a harmonic division.

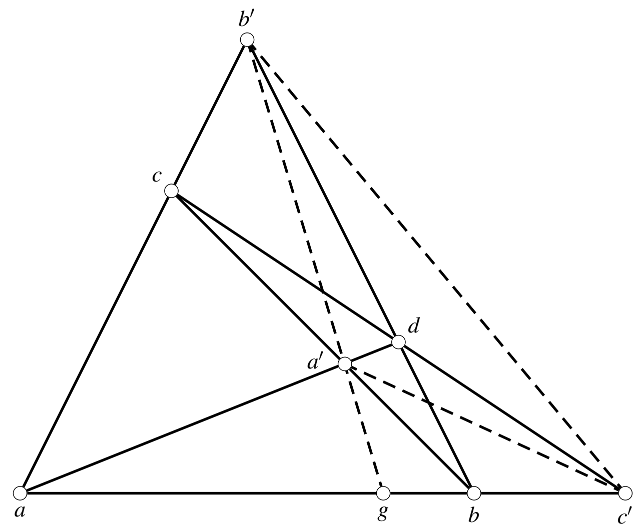


Fig. 5.8 A quadrangle, and harmonic divisions.

In fact, it can be shown that the following quadruples of lines form harmonic divisions: $(\langle c,a\rangle,\langle b',a'\rangle,\langle d,b\rangle,\langle b',c'\rangle)$, $(\langle b,a\rangle,\langle c',a'\rangle,\langle d,c\rangle,\langle c',b'\rangle)$, and $(\langle b,c\rangle, \langle a',c'\rangle,\langle a,d\rangle,\langle a',b'\rangle)$; see Figure 5.8. For more on harmonic divisions, the interested reader should consult any text on projective geometry (for example, Berger [3, 4], Samuel [23], Sidler [24], Tisseron [26], or Pedoe [21]).

Having the notion of cross-ratio at our disposal, we can interpret linear interpolation in the homogenization $\widehat{E}$ of an affine space E as determining a cross-ratio in the projective completion $\widetilde{E}$ of E! This simple fact provides a geometric interpretation of the rational version of the de Casteljau algorithm; see the additional material on the web site (see `http://www.cis.upenn.edu/~jean/gbooks/geom2.html`).

Given any affine space E, let θ_1 and θ_2 be two linearly independent vectors in $\widehat{E}$, and let $t\in K$ be any scalar. Consider

$$\theta_3 = \theta_1 \mathbin{\widehat{+}} \theta_2$$

and

$$\theta_4 = (1-t)\cdot\theta_1 \mathbin{\widehat{+}} t\cdot\theta_2.$$

Observe that the conditions for applying Lemma 5.13 are satisfied, and that the cross-ratio of the points $p(\theta_1)$, $p(\theta_2)$, $p(\theta_3)$, and $p(\theta_4)$ in the projective space $\widetilde{E}$ is given by

$$[p(\theta_1), p(\theta_2), p(\theta_3), p(\theta_4)] = [1-t, t]_{\sim}.$$

Assuming $t \neq 0$ (the case where $\theta_4 \neq \theta_2$), this yields

$$[p(\theta_1), p(\theta_2), p(\theta_3), p(\theta_4)] = \frac{1-t}{t}.$$

Thus, determining θ_4 using the affine interpolation

$$\theta_4 = (1-t)\cdot\theta_1 \widehat{+}\, t\cdot\theta_2$$

in $\widehat{E}$ is equivalent to finding the point $p(\theta_4)$ in the projective space $\widetilde{E}$ such that the cross-ratio of the four points $(p(\theta_1), p(\theta_2), p(\theta_3), p(\theta_4))$ is equal to $(1-t)/t$. In the particular case where $\theta_1 = \langle a, \alpha\rangle$ and $\theta_2 = \langle b, \beta\rangle$, where a and b are distinct points of E, if $\alpha+\beta \neq 0$ and $(1-t)\alpha + t\beta \neq 0$, we know that

$$\theta_3 = \left\langle \frac{\alpha}{\alpha+\beta}\, a + \frac{\beta}{\alpha+\beta}\, b,\ \alpha+\beta \right\rangle$$

and

$$\theta_4 = \left\langle \frac{(1-t)\alpha}{(1-t)\alpha+t\beta}\, a + \frac{t\beta}{(1-t)\alpha+t\beta}\, b,\ (1-t)\alpha+t\beta \right\rangle,$$

and letting

$$c = \frac{\alpha}{\alpha+\beta}\, a + \frac{\beta}{\alpha+\beta}\, b$$

and

$$d = \frac{(1-t)\alpha}{(1-t)\alpha+t\beta}\, a + \frac{t\beta}{(1-t)\alpha+t\beta}\, b,$$

we also have

$$[a, b, c, d] = \frac{1-t}{t}.$$

Readers may have fun in verifying that when $t = \frac{2}{3}$, the points (a, d, b, c) form a harmonic division!

When $\alpha+\beta = 0$ or $(1-t)\alpha + t\beta = 0$, we have to consider points at infinity, which is better handled in $\widetilde{E}$. In any case, the computation of d can be viewed as determining the unique point d such that $[a, b, c, d] = (1-t)/t$, using

$$c = \frac{\alpha}{\alpha+\beta}\, a + \frac{\beta}{\alpha+\beta}\, b.$$

5.9 Duality in Projective Geometry

We now consider duality in projective geometry. Given a vector space E of finite dimension $n+1$, recall that its *dual space* E^* is the vector space of all linear forms $f\colon E \to K$ and that E^* is isomorphic to E. We also have a canonical isomorphism between E and its bidual E^{**}, which allows us to identify E and E^{**}.

Let $\mathscr{H}(E)$ denote the set of hyperplanes in $\mathbf{P}(E)$. In Section 5.3 we observed that the map

$$p(f) \mapsto \mathbf{P}(\operatorname{Ker} f)$$

is a bijection between $\mathbf{P}(E^*)$ and $\mathscr{H}(E)$, in which the equivalence class $p(f) = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ is mapped to the hyperplane $\mathbf{P}(\operatorname{Ker} f)$. Using the above bijection between $\mathbf{P}(E^*)$ and $\mathscr{H}(E)$, a projective subspace $\mathbf{P}(U)$ of $\mathbf{P}(E^*)$ (where U is a subspace of E^*) can be identified with a subset of $\mathscr{H}(E)$, namely the family

$$\{\mathbf{P}(H) \mid H = \operatorname{Ker} f,\, f \in U - \{0\}\}$$

consisting of the projective hyperplanes in $\mathscr{H}(E)$ corresponding to nonnull linear forms in U. Such subsets of $\mathscr{H}(E)$ are called *linear systems (of hyperplanes)*.

The bijection between $\mathbf{P}(E^*)$ and $\mathscr{H}(E)$ allows us to view $\mathscr{H}(E)$ as a projective space, and linear systems as projective subspaces of $\mathscr{H}(E)$. In the projective space $\mathscr{H}(E)$, a point is a hyperplane in $\mathbf{P}(E)$! The duality between subspaces of E and subspaces of E^* (reviewed below) and the fact that there is a bijection between $\mathbf{P}(E^*)$ and $\mathscr{H}(E)$ yields a powerful duality between the set of projective subspaces of $\mathbf{P}(E)$ and the set of linear systems in $\mathscr{H}(E)$ (or equivalently, the set of projective subspaces of $\mathbf{P}(E^*)$).

The idea of duality in projective geometry goes back to Gergonne and Poncelet, in the early nineteenth century. However, Poncelet had a more restricted type of duality in mind (polarity with respect to a conic or a quadric), whereas Gergonne had the more general idea of the duality between points and lines (or points and planes). This more general duality arises from a specific pairing between E and E^* (a nonsingular bilinear form). Here we consider the pairing $\langle -,- \rangle\colon E^* \times E \to K$, defined such that

$$\langle f, v \rangle = f(v),$$

for all $f \in E^*$ and all $v \in E$. Recall that given a subset V of E (respectively a subset U of E^*), the *orthogonal* V^0 *of* V is the subspace of E^* defined such that

$$V^0 = \{f \in E^* \mid \langle f, v \rangle = 0, \text{ for every } v \in V\},$$

and that the *orthogonal* U^0 *of* U is the subspace of E defined such that

$$U^0 = \{v \in E \mid \langle f, v \rangle = 0, \text{ for every } f \in U\}.$$

Then, by a standard theorem (since E and E^* have the same finite dimension $n+1$), $U = U^{00}$, $V = V^{00}$, and the maps

$$V \mapsto V^0 \quad \text{and} \quad U \mapsto U^0$$

are inverse bijections, where V is a subspace of E, and U is a subspace of E^*.

These maps set up a *duality* between subspaces of E and subspaces of E^*. Furthermore, we know that U has dimension k iff U^0 has dimension $n+1-k$, and similarly for V and V^0.

Since a linear system $P = \mathbf{P}(U)$ of hyperplanes in $\mathscr{H}(E)$ corresponds to a subspace U of E^*, and since U^0 is the intersection of all the hyperplanes defined by nonnull linear forms in U, we can view a linear system $P = \mathbf{P}(U)$ in $\mathscr{H}(E)$ as the family of hyperplanes containing $\mathbf{P}(U^0)$.

In view of the identification of $\mathbf{P}(E^*)$ with the set $\mathscr{H}(E)$ of hyperplanes in $\mathbf{P}(E)$, by passing to projective spaces, the above bijection between the set of subspaces of E and the set of subspaces of E^* yields a bijection between the set of projective subspaces of $\mathbf{P}(E)$ and the set of linear systems in $\mathscr{H}(E)$ (or equivalently, the set of projective subspaces of $\mathbf{P}(E^*)$).

More specifically, assuming that E has dimension $n+1$, so that $\mathbf{P}(E)$ has dimension n, if $Q = \mathbf{P}(V)$ is any projective subspace of $\mathbf{P}(E)$ (where V is any subspace of E) and if $P = \mathbf{P}(U)$ is any linear system in $\mathscr{H}(E)$ (where U is any subspace of E^*), we get a subspace Q^0 of $\mathscr{H}(E)$ defined by

$$Q^0 = \{\mathbf{P}(H) \mid Q \subseteq \mathbf{P}(H),\, \mathbf{P}(H) \text{ a hyperplane in } \mathscr{H}(E)\},$$

and a subspace P^0 of $\mathbf{P}(E)$ defined by

$$P^0 = \bigcap \{\mathbf{P}(H) \mid \mathbf{P}(H) \in P,\, \mathbf{P}(H) \text{ a hyperplane in } \mathscr{H}(E)\}.$$

We have $P = P^{00}$ and $Q = Q^{00}$. Since Q^0 is determined by $\mathbf{P}(V^0)$, if $Q = \mathbf{P}(V)$ has dimension k (i.e., if V has dimension $k+1$), then Q^0 has dimension $n-k-1$ (since V has dimension $k+1$ and $\dim(E) = n+1$, then V^0 has dimension $n+1-(k+1) = n-k$). Thus,

$$\dim(Q) + \dim(Q^0) = n - 1,$$

and similarly, $\dim(P) + \dim(P^0) = n - 1$.

A linear system $P = \mathbf{P}(U)$ of hyperplanes in $\mathscr{H}(E)$ is called a *pencil of hyperplanes* if it corresponds to a projective line in $\mathbf{P}(E^*)$, which means that U is a subspace of dimension 2 of E^*. From $\dim(P) + \dim(P^0) = n - 1$, a pencil of hyperplanes P is the family of hyperplanes in $\mathscr{H}(E)$ containing some projective subspace $\mathbf{P}(V)$ of dimension $n-2$ (where $\mathbf{P}(V)$ is a projective subspace of $\mathbf{P}(E)$, and $\mathbf{P}(E)$ has dimension n). When $n = 2$, a pencil of hyperplanes in $\mathscr{H}(E)$, also called a *pencil of lines*, is the family of lines passing through a given point. When $n = 3$, a pencil of hyperplanes in $\mathscr{H}(E)$, also called a *pencil of planes*, is the family of planes passing through a given line.

When $n = 2$, the above duality takes a rather simple form. In this case (of a projective plane $\mathbf{P}(E)$), the duality is a bijection between points and lines with the following properties:

- A point a maps to a line D_a (the pencil of lines in $\mathscr{H}(E)$ containing a, also denoted by a^*)
- A line D maps to a point p_D (the line D in $\mathscr{H}(E)$!)
- Two points a,b map to lines D_a, D_b, such that the intersection of D_a and D_b is the point $p_{\langle a,b\rangle}$ corresponding to the line $\langle a,b\rangle$ via duality
- A line D containing two points a,b maps to the intersection p_D of the lines D_a and D_b.
- If $a \in D$, where a is a point and D is a line, then $p_D \in D_a$.

The reader will discover that the dual of Desargues's theorem is its converse. This is a nice way of getting the converse for free! We will not spoil the reader's fun and let him discover the dual of Pappus's theorem.

To conclude our quick tour of projective geometry, we estabish a connection between the cross-ratio of hyperplanes in a pencil of hyperplanes with the cross-ratio of the intersection points of any line not contained in any hyperplane in this pencil with four hyperplanes in this pencil.

5.10 Cross-Ratios of Hyperplanes

Given a pencil $P = \mathbf{P}(U)$ of hyperplanes in $\mathscr{H}(E)$, for any sequence (H_1, H_2, H_3, H_4) of hyperplanes in this pencil, if H_1, H_2, H_3 are distinct, we define the cross-ratio $[H_1, H_2, H_3, H_4]$ as the cross-ratio of the hyperplanes H_i considered as points on the projective line P in $\mathbf{P}(E^*)$. In particular, in a projective plane $\mathbf{P}(E)$, given any four concurrent lines D_1, D_2, D_3, D_4, where D_1, D_2, D_3 are distinct, for any two distinct lines Δ and Δ' not passing through the common intersection c of the lines D_i, letting $d_i = \Delta \cap D_i$, and $d_i' = \Delta' \cap D_i$, note that the projection of center c from Δ to Δ' maps each d_i to d_i'.

Since such a projection is a projectivity, and since projectivities between lines preserve cross-ratios, we have

$$[d_1, d_2, d_3, d_4] = [d_1', d_2', d_3', d_4'],$$

which means that the cross-ratio of the d_i is independent of the line Δ (see Figure 5.9).

In fact, this cross-ratio is equal to $[D_1, D_2, D_3, D_4]$, as shown in the next lemma.

Lemma 5.14. *Let $P = \mathbf{P}(U)$ be a pencil of hyperplanes in $\mathscr{H}(E)$, and let $\Delta = \mathbf{P}(D)$ be any projective line such that $\Delta \notin H$ for all $H \in P$. The map $h\colon P \to \Delta$ defined such that $h(H) = H \cap \Delta$ for every hyperplane $H \in P$ is a projectivity. Furthermore, for any sequence (H_1, H_2, H_3, H_4) of hyperplanes in the pencil P, if H_1, H_2, H_3 are distinct and $d_i = \Delta \cap H_i$, then $[d_1, d_2, d_3, d_4] = [H_1, H_2, H_3, H_4]$.*

Proof. First, the map $h\colon P \to \Delta$ is well–defined, since in a projective space, every line $\Delta = \mathbf{P}(D)$ not contained in a hyperplane intersects this hyperplane in exactly one point. Since $P = \mathbf{P}(U)$ is a pencil of hyperplanes in $\mathscr{H}(E)$, U has dimension 2,

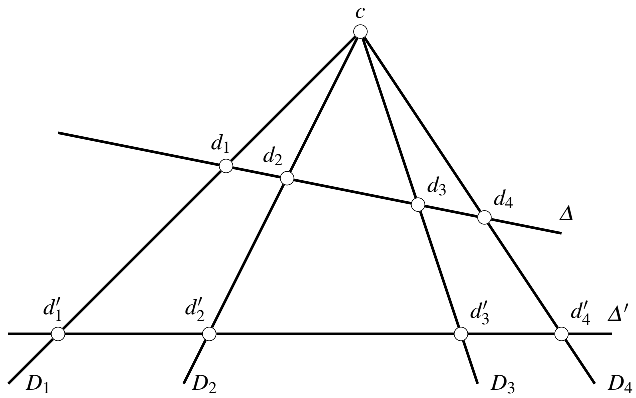


Fig. 5.9 A pencil of lines and its cross-ratio with intersecting lines.

and let φ and ψ be two nonnull linear forms in E^* that constitute a basis of U, and let $F = \varphi^{-1}(0)$ and $G = \psi^{-1}(0)$. Let $a = \mathbf{P}(F) \cap \Delta$ and $b = \mathbf{P}(G) \cap \Delta$. There are some vectors $u, v \in D$ such that $a = p(u)$ and $b = p(v)$, and since φ and ψ are linearly independent, we have $a \neq b$, and we can choose φ and ψ such that $\varphi(v) = -1$ and $\psi(u) = 1$. Also, (u, v) is a basis of D. Then a point $p(\alpha u + \beta v)$ on Δ belongs to the hyperplane $H = p(\gamma\varphi + \delta\psi)$ of the pencil P iff

$$(\gamma\varphi + \delta\psi)(\alpha u + \beta v) = 0,$$

which, since $\varphi(u) = 0$, $\psi(v) = 0$, $\varphi(v) = -1$, and $\psi(u) = 1$, yields $\gamma\beta = \delta\alpha$, which is equivalent to $[\alpha, \beta] = [\gamma, \delta]$ in $\mathbf{P}(K^2)$. But then the map $h \colon P \to \Delta$ is a projectivity. Letting $d_i = \Delta \cap H_i$, since by Lemma 5.12 a projectivity of lines preserves the cross-ratio, we get $[d_1, d_2, d_3, d_4] = [H_1, H_2, H_3, H_4]$. □

5.11 Complexification of a Real Projective Space

Notions such as orthogonality, angles, and distance between points are not projective concepts. In order to define such notions, one needs an inner product on the underlying vector space. We say that such notions belong to *Euclidean geometry*. At first glance, the fact that some important Euclidean concepts are not covered by projective geometry seems a major drawback of projective geometry. Fortunately, geometers of the nineteenth century (including Laguerre, Monge, Poncelet, Chasles, von Staudt, Cayley, and Klein) found an astute way of recovering certain Euclidean notions such as angles and orthogonality (also circles) by embedding real projective spaces into complex projective spaces. In the next two sections we will give a brief account of this method. More details can be found in Berger [3, 4], Pedoe [21], Samuel [23], Coxeter [5, 6], Sidler [24], Tisseron [26], Lehmann and Bkouche [20], and, of course, Volume II of Veblen and Young [29]. Readers may want to consult

Chapter 6, which gives a review of Euclidean geometry, especially Section 8.8, on angles.

The first step is to embed a real vector space E into a complex vector space $E_{\mathbb{C}}$. A quick but somewhat bewildering way to do so is to define the complexification of E as the tensor product $\mathbb{C} \otimes E$. A more tangible way is to define the following structure.

Definition 5.10. Given a real vector space E, let $E_{\mathbb{C}}$ be the structure $E \times E$ under the addition operation

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2),$$

and let multiplication by a complex scalar $z = x + \mathrm{i}y$ be defined such that

$$(x + \mathrm{i}y) \cdot (u, v) = (xu - yv, yu + xv).$$

It is easily shown that the structure $E_{\mathbb{C}}$ is a complex vector space. It is also immediate that

$$(0, v) = \mathrm{i}(v, 0),$$

and thus, identifying E with the subspace of $E_{\mathbb{C}}$ consisting of all vectors of the form $(u, 0)$, we can write

$$(u, v) = u + \mathrm{i}v.$$

Given a vector $w = u + \mathrm{i}v$, its *conjugate* $\overline{w}$ is the vector $\overline{w} = u - \mathrm{i}v$. Then conjugation is a map from $E_{\mathbb{C}}$ to itself that is an involution. If $(e_1, \ldots, e_n)$ is any basis of E, then $((e_1, 0), \ldots, (e_n, 0))$ is a basis of $E_{\mathbb{C}}$. We call such a basis a *real basis*.

Given a linear map $f\colon E \to E$, the map f can be extended to a linear map $f_{\mathbb{C}}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ defined such that

$$f_{\mathbb{C}}(u + \mathrm{i}v) = f(u) + \mathrm{i}f(v).$$

We define the *complexification* of $\mathbf{P}(E)$ as $\mathbf{P}(E_{\mathbb{C}})$. If $\left(E, \overrightarrow{E}\right)$ is a real affine space, we define the *complexified projective completion of* $\left(E, \overrightarrow{E}\right)$ as $\mathbf{P}(\widehat{E}_{\mathbb{C}})$ and denote it by $\widetilde{E}_{\mathbb{C}}$. Then $\widetilde{E}$ is naturally embedded in $\widetilde{E}_{\mathbb{C}}$, and it is called the set of *real points* of $\widetilde{E}_{\mathbb{C}}$.

If E has dimension $n + 1$ and $(e_1, \ldots, e_{n+1})$ is a basis of E, given any homogeneous polynomial $P(x_1, \ldots, x_{n+1})$ over $\mathbb{C}$ of total degree m, because P is homogeneous, it is immediately verified that

$$P(x_1, \ldots, x_{n+1}) = 0$$

iff

$$P(\lambda x_1, \ldots, \lambda x_{n+1}) = 0,$$

for any $\lambda \neq 0$. Thus, we can define the *hypersurface* $V(P)$ *of equation* $P(x_1, \ldots, x_{n+1}) = 0$ as the subset of $\widetilde{E}_{\mathbb{C}}$ consisting of all points of homogeneous coordinates $(x_1, \ldots, x_{n+1})$ such that $P(x_1, \ldots, x_{n+1}) = 0$. We say that the hypersurface $V(P)$

of equation $P(x_1,\dots,x_{n+1}) = 0$ is *real* whenever $P(x_1,\dots,x_{n+1}) = 0$ implies that $P(\overline{x}_1,\dots,\overline{x}_{n+1}) = 0$.

Note that a real hypersurface may have points other than real points, or no real points at all. For example,

$$x^2 + y^2 - z^2 = 0$$

contains real and complex points such as $(1,\mathrm{i},0)$ and $(1,-\mathrm{i},0)$, and

$$x^2 + y^2 + z^2 = 0$$

contains only complex points. When $m = 2$ (where m is the total degree of P), a hypersurface is called a *quadric*, and when $m = 2$ and $n = 2$, a *conic*. When $m = 1$, a hypersurface is just a hyperplane.

Given any homogeneous polynomial $P(x_1,\dots,x_{n+1})$ over $\mathbb{R}$ of total degree m, since $\mathbb{R} \subseteq \mathbb{C}$, P viewed as a homogeneous polynomial over $\mathbb{C}$ defines a hypersurface $V(P)_{\mathbb{C}}$ in $\widetilde{E}_{\mathbb{C}}$, and also a hypersurface $V(P)$ in $\mathbf{P}(E)$. It is clear that $V(P)$ is naturally embedded in $V(P)_{\mathbb{C}}$, and $V(P)_{\mathbb{C}}$ is called the *complexification of* $V(P)$.

We now show how certain real quadrics without real points can be used to define orthogonality and angles.

5.12 Similarity Structures on a Projective Space

We begin with a real Euclidean plane $\left(E, \overrightarrow{E}\right)$. We will show that the angle of two lines D_1 and D_2 can be expressed as a certain cross-ratio involving the lines D_1, D_2 and also two lines D_I and D_J joining the intersection point $D_1 \cap D_2$ of D_1 and D_2 to two complex points at infinity I and J called the *circular points*. However, there is a slight problem, which is that we haven't yet defined the angle of two lines! Recall from Section 8.8 that we define the (oriented) angle $\widehat{u_1 u_2}$ of two unit vectors u_1, u_2 as the equivalence class of pairs of unit vectors under the equivalence relation defined such that

$$\langle u_1, u_2\rangle \equiv \langle u_3, u_4\rangle$$

iff there is some rotation r such that $r(u_1) = u_3$ and $r(u_2) = u_4$. The set of (oriented) angles of vectors is a group isomorphic to the group $\mathbf{SO}(2)$ of plane rotations. If the Euclidean plane is oriented, the measure of the angle of two vectors is defined up to $2k\pi$ ($k \in \mathbb{Z}$). The angle of two vectors has a measure that is either θ or $2\pi - \theta$, where $\theta \in [0, 2\pi[$, depending on the orientation of the plane. The problem with lines is that they are not oriented: A line is defined by a point a and a vector u, but also by a and $-u$. Given any two lines D_1 and D_2, if r is a rotation of angle θ such that $r(D_1) = D_2$, note that the rotation $-r$ of angle $\theta + \pi$ also maps D_1 onto D_2. Thus, in order to define the (oriented) angle $\widehat{D_1 D_2}$ of two lines D_1, D_2, we define an equivalence relation on pairs of lines as follows:

$$\langle D_1, D_2 \rangle \equiv \langle D_3, D_4 \rangle$$

if there is some rotation r such that $r(D_1) = D_2$ and $r(D_3) = D_4$.

It can be verified that the set of (oriented) angles of lines is a group isomorphic to the quotient group $\mathbf{SO}(2)/\{\mathrm{id}, -\mathrm{id}\}$, also denoted by $\mathbf{PSO}(2)$. In order to define the measure of the angle of two lines, the Euclidean plane E must be oriented. The measure of the angle $\widehat{D_1 D_2}$ of two lines is defined up to $k\pi$ ($k \in \mathbb{Z}$). The angle of two lines has a measure that is either θ or $\pi - \theta$, where $\theta \in [0, \pi[$, depending on the orientation of the plane. We now go back to the circular points.

Let (a_0, a_1, a_2, a_3) be any projective frame for $\widetilde{E}_{\mathbb{C}}$ such that (a_0, a_1) arises from an orthonormal basis (u_1, u_2) of $\overrightarrow{E}$ and the line at infinity H corresponds to $z = 0$ (where (x, y, z) are the homogeneous coordinates of a point w.r.t. (a_0, a_1, a_2, a_3)). Consider the points belonging to the intersection of the real conic Σ of equation

$$x^2 + y^2 - z^2 = 0$$

with the line at infinity $z = 0$. For such points, $x^2 + y^2 = 0$ and $z = 0$, and since

$$x^2 + y^2 = (y - \mathrm{i}x)(y + \mathrm{i}x),$$

we get exactly two points I and J of homogeneous coordinates $(1, -\mathrm{i}, 0)$ and $(1, \mathrm{i}, 0)$. The points I and J are called the *circular points*, or the *absolute points*, of $\widetilde{E}_{\mathbb{C}}$. They are complex points at infinity. Any line containing either I or J is called an *isotropic line*.

What is remarkable about I and J is that they allow the definition of the angle of two lines in terms of a certain cross-ratio. Indeed, consider two distinct real lines D_1 and D_2 in E, and let D_I and D_J be the isotropic lines joining $D_1 \cap D_2$ to I and J. We will compute the cross-ratio $[D_1, D_2, D_I, D_J]$. For this, we simply have to compute the cross-ratio of the four points obtained by intersecting D_1, D_2, D_I, D_J with any line not passing through $D_1 \cap D_2$. By changing frame if necessary, so that $D_1 \cap D_2 = a_0$, we can assume that the equations of the lines D_1, D_2, D_I, D_J are of the form

$$\begin{aligned} y &= m_1 x, \\ y &= m_2 x, \\ y &= -\mathrm{i}x, \\ y &= \mathrm{i}x, \end{aligned}$$

leaving the cases $m_1 = \infty$ and $m_2 = \infty$ as a simple exercise. If we choose $z = 1$ as the intersecting line, we need to compute the cross-ratio of the points $(D_1)_\infty = (1, m_1, 0)$, $(D_2)_\infty = (1, m_2, 0)$, $I = (1, -\mathrm{i}, 0)$, and $J = (1, \mathrm{i}, 0)$, and we get

$$[D_1, D_2, D_I, D_J] = [(D_1)_\infty, (D_2)_\infty, I, J] = \frac{(-\mathrm{i} - m_1)}{(\mathrm{i} - m_1)} \frac{(\mathrm{i} - m_2)}{(-\mathrm{i} - m_2)},$$

that is,

$$[D_1, D_2, D_I, D_J] = \frac{m_1 m_2 + 1 + \mathrm{i}(m_2 - m_1)}{m_1 m_2 + 1 - \mathrm{i}(m_2 - m_1)}.$$

However, since m_1 and m_2 are the slopes of the lines D_1 and D_2, it is well known that if θ is the (oriented) angle between D_1 and D_2, then

$$\tan\theta = \frac{m_2 - m_1}{m_1 m_2 + 1}.$$

Thus, we have

$$[D_1, D_2, D_I, D_J] = \frac{m_1 m_2 + 1 + \mathrm{i}(m_2 - m_1)}{m_1 m_2 + 1 - \mathrm{i}(m_2 - m_1)} = \frac{1 + \mathrm{i}\tan\theta}{1 - \mathrm{i}\tan\theta},$$

that is,

$$[D_1, D_2, D_I, D_J] = \cos 2\theta + \mathrm{i}\sin 2\theta = \mathrm{e}^{\mathrm{i}2\theta}.$$

One can check that the formula still holds when $m_1 = \infty$ or $m_2 = \infty$, and also when $D_1 = D_2$. The formula

$$[D_1, D_2, D_I, D_J] = \mathrm{e}^{\mathrm{i}2\theta}$$

is known as *Laguerre's formula*.

If U denotes the group $\{\mathrm{e}^{\mathrm{i}\theta} \mid -\pi \le \theta \le \pi\}$ of complex numbers of modulus 1, recall that the map $\Lambda\colon \mathbb{R} \to U$ defined such that

$$\Lambda(t) = \mathrm{e}^{\mathrm{i}t}$$

is a group homomorphism such that $\Lambda^{-1}(1) = 2k\pi$, where $k \in \mathbb{Z}$. The restriction

$$\Lambda\colon\ ]-\pi, \pi[\to (U - \{-1\})$$

of Λ to $]-\pi, \pi[$ is a bijection, and its inverse will be denoted by

$$\log_U\colon (U - \{-1\}) \to]-\pi, \pi[.$$

For stating Lemma 5.15 more conveniently, we will extend $\log_U$ to U by letting $\log_U(-1) = \pi$, even though the resulting function is not continuous at -1!. Then we can write

$$\theta = \frac{1}{2}\log_U([D_1, D_2, D_I, D_J]).$$

If the orientation of the plane E is reversed, θ becomes $\pi - \theta$, and since

$$\mathrm{e}^{\mathrm{i}2(\pi-\theta)} = \mathrm{e}^{2\mathrm{i}\pi - \mathrm{i}2\theta} = \mathrm{e}^{-\mathrm{i}2\theta},$$

$\log_U(\mathrm{e}^{\mathrm{i}2(\pi-\theta)}) = -\log_U(\mathrm{e}^{\mathrm{i}2\theta})$, and

$$\theta = -\frac{1}{2}\log_U([D_1, D_2, D_I, D_J]).$$

In all cases, we have

$$\theta = \frac{1}{2}|\log_U([D_1, D_2, D_I, D_J])|,$$

a formula due to Cayley. We summarize the above in the following lemma.

Lemma 5.15. *Given any two lines D_1, D_2 in a real Euclidean plane $(E, \overrightarrow{E})$, letting D_I and D_J be the isotropic lines in $\widetilde{E}_{\mathbb{C}}$ joining the intersection point $D_1 \cap D_2$ of D_1 and D_2 to the circular points I and J, if θ is the angle of the two lines D_1, D_2, we have*

$$[D_1, D_2, D_I, D_J] = \mathrm{e}^{\mathrm{i}2\theta},$$

known as Laguerre's formula, and independently of the orientation of the plane, we have

$$\theta = \frac{1}{2}|\log_U([D_1, D_2, D_I, D_J])|,$$

known as Cayley's formula.

In particular, note that $\theta = \pi/2$ iff $[D_1, D_2, D_I, D_J] = -1$, that is, if (D_1, D_2, D_I, D_J) forms a harmonic division. Thus, two lines D_1 and D_2 are orthogonal iff they form a harmonic division with D_I and D_J.

The above considerations show that it is not necessary to assume that $(E, \overrightarrow{E})$ is a real Euclidean plane to define the angle of two lines and orthogonality. Instead, it is enough to assume that two complex conjugate points I, J on the line H at infinity are given. We say that $\langle I, J\rangle$ provides a *similarity structure* on $\widetilde{E}_{\mathbb{C}}$. Note in passing that a circle can be defined as a conic in $\widetilde{E}_{\mathbb{C}}$ that contains the circular points I, J. Indeed, the equation of a conic is of the form

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0.$$

If this conic contains the circular points $I = (1, -\mathrm{i}, 0)$ and $J = (1, \mathrm{i}, 0)$, we get the two equations

$$a - b - \mathrm{i}c = 0,$$
$$a - b + \mathrm{i}c = 0,$$

from which we get $2\mathrm{i}c = 0$ and $a = b$, that is, $c = 0$ and $a = b$. The resulting equation

$$ax^2 + ay^2 + dxz + eyz + fz^2 = 0$$

is indeed that of a circle.

Instead of using the function $\log_U\colon (U - \{-1\}) \to]-\pi, \pi[$ as logarithm, one may use the complex logarithm function $\log\colon \mathbb{C}^* \to B$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$ and

$$B = \{x + \mathrm{i}y \mid x, y \in \mathbb{R}, -\pi < y \leq \pi\}.$$

Indeed, the restriction of the complex exponential function $z \mapsto \mathrm{e}^z$ to B is bijective, and thus, log is well-defined on C^* (note that log is a homeomorphism from $\mathbb{C} - \{x \mid$

$x \in \mathbb{R}, x \leq 0\}$ onto $\{x + iy \mid x, y \in \mathbb{R}, -\pi < y < \pi\}$, the interior of B). Then Cayley's formula reads as

$$\theta = \frac{1}{2\mathrm{i}} \log([D_1, D_2, D_I, D_J]),$$

with a $\pm$ in front when the plane is nonoriented. Observe that this formula allows the definition of the angle of two complex lines (possibly a complex number) and the notion of orthogonality of complex lines. In this case, note that the isotropic lines are orthogonal to themselves!

The definition of orthogonality of two lines D_1, D_2 in terms of (D_1, D_2, D_I, D_J) forming a harmonic division can be used to give elegant proofs of various results. Cayley's formula can even be used in computer vision to explain modeling and calibrating cameras! (see Faugeras [10]). As an illustration, consider a triangle (a, b, c), and recall that the line a' passing through a and orthogonal to (b, c) is called the *altitude of* a, and similarly for b and c. It is well known that the altitudes a', b', c' intersect in a common point called the *orthocenter* of the triangle (a, b, c). This can be shown in a number of ways using the circular points. Indeed, letting bc_∞, ab_∞, ac_∞, a'_∞, b'_∞, and c'_∞ denote the points at infinity of the lines $\langle b, c\rangle$, $\langle a, b\rangle$, $\langle a, c\rangle$, a', b', and c', we have

$$[bc_\infty, a'_\infty, I, J] = -1, \quad [ab_\infty, c'_\infty, I, J] = -1, \quad [ac_\infty, b'_\infty, I, J] = -1,$$

and it is easy to show that there is an involution σ of the line at infinity such that

$$\begin{aligned} \sigma(I) &= J, \\ \sigma(J) &= I, \\ \sigma(bc_\infty) &= a'_\infty, \\ \sigma(ab_\infty) &= c'_\infty, \\ \sigma(ac_\infty) &= b'_\infty. \end{aligned}$$

Then, using the result stated in Problem 5.28, the lines a', b', c' are concurrent. For more details and other results, notably on the conics, see Sidler [24], Berger [4], and Samuel [23].

The generalization of what we just did to real Euclidean spaces $\left(E, \overrightarrow{E}\right)$ of dimension n is simple. Let $(a_0, \ldots, a_{n+1})$ be any projective frame for $\widetilde{E}_{\mathbb{C}}$ such that $(a_0, \ldots, a_{n-1})$ arises from an orthonormal basis $(u_1, \ldots, u_n)$ of $\overrightarrow{E}$ and the hyperplane at infinity H corresponds to $x_{n+1} = 0$ (where $(x_1, \ldots, x_{n+1})$ are the homogeneous coordinates of a point with respect to $(a_0, \ldots, a_{n+1})$). Consider the points belonging to the intersection of the real quadric Σ of equation

$$x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 0$$

with the hyperplane at infinity $x_{n+1} = 0$. For such points,

$$x_1^2 + \cdots + x_n^2 = 0 \quad \text{and} \quad x_{n+1} = 0.$$

Such points belong to a quadric called the *absolute quadric* of $\widetilde{E}_{\mathbb{C}}$, and denoted by Ω. Any line containing any point on the absolute quadric is called an *isotropic line*. Then, given any two coplanar lines D_1 and D_2 in E, these lines intersect the hyperplane at infinity H in two points $(D_1)_\infty$ and $(D_2)_\infty$, and the line Δ joining $(D_1)_\infty$ and $(D_2)_\infty$ intersects the absolute quadric Ω in two conjugate points I_Δ and J_Δ (also called circular points). It can be shown that the angle θ between D_1 and D_2 is defined by Laguerre's formula:

$$[(D_1)_\infty,(D_2)_\infty,I_\Delta,J_\Delta] = [D_1,D_2,D_{I_\Delta},D_{J_\Delta}] = \mathrm{e}^{\mathrm{i}2\theta},$$

where D_{I_Δ} and D_{J_Δ} are the lines joining the intersection $D_1 \cap D_2$ of D_1 and D_2 to the circular points I_Δ and J_Δ.

As in the case of a plane, the above considerations show that it is not necessary to assume that $\left(E, \overrightarrow{E}\right)$ is a real Euclidean space to define the angle of two lines and orthogonality. Instead, it is enough to assume that a nondegenerate real quadric Ω in the hyperplane at infinity H and without real points is given. In particular, when $n = 3$, the absolute quadric Ω is a nondegenerate real conic consisting of complex points at infinity. We say that Ω provides a *similarity structure* on $\widetilde{E}_{\mathbb{C}}$.

It is also possible to show that the projectivities of $\widetilde{E}_{\mathbb{C}}$ that leave both the hyperplane H at infinity and the absolute quadric Ω (globally) invariant form a group which is none other than the group of similarities. A *similarity* is a map that is the composition of an isometry (a member of $\mathbf{O}(n)$), a central dilatation, and a translation. For more details on the use of absolute quadrics to obtain some very sophisticated results, the reader should consult Berger [3, 4], Pedoe [21], Samuel [23], Coxeter [5], Sidler [24], Tisseron [26], Lehmann and Bkouche [20], and, of course, Volume II of Veblen and Young [29], which also explains how some non-Euclidean geometries are obtained by chosing the absolute quadric in an appropriate fashion (after Cayley and Klein).

5.13 Some Applications of Projective Geometry

Projective geometry is definitely a jewel of pure mathematics and one of the major mathematical achievements of the nineteenth century. It turns out to be a prerequisite for algebraic geometry, but to our surprise (and pleasure), it also turns out to have applications in engineering. In this short section we summarize some of these applications.

We first discuss applications of projective geometry to camera calibration, a crucial problem in computer vision. Our brief presentation follows quite closely Trucco and Verri [27] (Chapter 2 and Chapter 6). One should also consult Faugeras [10], or Jain, Katsuri, and Schunck [18].

The *pinhole* (or *perspective*) model of a camera is a typical example from computer vision that can be explained very simply in terms of projective transformations. A pinhole camera consists of a point $\mathbf{O}$ called the *center* or *focus of projection*, and a plane π (not containing $\mathbf{O}$) called the *image plane*. The distance f from the image plane π to the center $\mathbf{O}$ is called the *focal length*. The line through $\mathbf{O}$ and perpendicular to π is called the *optical axis*, and the point $\mathbf{o}$, intersection of the optical axis with the image plane is called the *principal point* or *image center*. The way the camera works is that a point P in 3D space is projected onto the image plane (the film) to a point p via the central projection of center $\mathbf{O}$.

It is assumed that an orthonormal frame $\mathscr{F}_c$ is attached to the camera, with its origin at $\mathbf{O}$ and its z-axis parallel to the optical axis. Such a frame is called the *camera reference frame*. With respect to the camera reference frame, it is very easy to write the equations relating the coordinates (x,y) (omitting $z=f$) of the image p (in the image plane π) of a point P of coordinates (X,Y,Z):

$$x = f\frac{X}{Z}, \quad y = f\frac{Y}{Z}.$$

Typically, points in 3D space are defined by their coordinates not with respect to the camera reference frame, but with respect to another frame $\mathscr{F}_w$, called the *world reference frame*. However, for most computer vision algorithms, it is necessary to know the coordinates of a point in 3D space with respect to the camera reference frame. Thus, it is necessary to know the position and orientation of the camera with respect to the frame $\mathscr{F}_w$. The position and orientation of the camera are given by some affine transformation $(R,\mathbf{T})$ mapping the frame $\mathscr{F}_w$ to the frame $\mathscr{F}_c$, where R is a rotation matrix and $\mathbf{T}$ is a translation vector. Furthermore, the coordinates of an image point are typically known in terms of *pixel coordinates*, and it is also necessary to transform the coordinates of an image point with respect to the camera reference frame to pixel coordinates. In summary, it is necessary to know the transformation that maps a point P in world coordinates (w.r.t. $\mathscr{F}_w$) to pixel coordinates.

This transformation of world coordinates to pixel coordinates turns out to be a projective transformation that depends on the extrinsic and the intrinsic parameters of the camera. The *extrinsic parameters* of a camera are the location and orientation of the camera with respect to the world reference frame $\mathscr{F}_w$. It is given by an affine map (in fact, a rigid motion, see Chapter 8, Section 8.4). The *intrinsic parameters* of a camera are the parameters needed to link the pixel coordinates of an image point to the corresponding coordinates in the camera reference frame. If $\mathbf{P_w} = (X_w, Y_w, Z_w)$ and $\mathbf{P_c} = (X_c, Y_c, Z_c)$ are the coordinates of the 3D point P with respect to the frames $\mathscr{F}_w$ and $\mathscr{F}_c$, respectively, we can write

$$\mathbf{P_c} = R(\mathbf{P_w} - \mathbf{T}).$$

Neglecting distorsions possibly introduced by the optics, the correspondence between the coordinates (x,y) of the image point with respect to $\mathscr{F}_c$ and the pixel coordinates $(x_{\mathrm{im}}, y_{\mathrm{im}})$ is given by

$$\begin{aligned} x &= -(x_{\mathrm{im}} - o_x)s_x, \\ y &= -(y_{\mathrm{im}} - o_y)s_y, \end{aligned}$$

where (o_x, o_y) are the pixel coordinates the principal point $\mathbf{o}$ and s_x, s_y are scaling parameters.

After some simple calculations, the upshot of all this is that the transformation between the homogeneous coordinates $(X_w, Y_w, Z_w, 1)$ of a 3D point and its homogeneous pixel coordinates (x_1, x_2, x_3) is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = M \begin{pmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{pmatrix}$$

where the matrix M, known as the *projection matrix*, is a 3×4 matrix depending on R, $\mathbf{T}$, o_x, o_y, f (the focal length), and s_x, s_y (for the derivation of this equation, see Trucco and Verri [27], Chapter 2).

The problem of estimating the extrinsic and the instrinsic parameters of a camera is known as the *camera calibration* problem. It is an important problem in computer vision. Now, using the equations

$$\begin{aligned} x &= -(x_{\text{im}} - o_x)s_x, \\ y &= -(y_{\text{im}} - o_y)s_y, \end{aligned}$$

we get

$$\begin{aligned} x_{\text{im}} &= -\frac{f}{s_x}\frac{X_c}{Z_c} + o_x, \\ y_{\text{im}} &= -\frac{f}{s_y}\frac{Y_c}{Z_c} + o_y, \end{aligned}$$

relating the coordinates w.r.t. the camera reference frame to the pixel coordinates. This suggests using the parameters $f_x = f/s_x$ and $f_y = f/s_y$ instead of the parameters f, s_x, s_y. In fact, all we need are the parameters $f_x = f/s_x$ and $\alpha = s_y/s_x$, called the *aspect ratio*. Without loss of generality, it can also be assumed that (o_x, o_y) are known. Then we have a total of eight parameters.

One way of solving the calibration problem is to try estimating f_x, α, the rotation matrix R, and the translation vector $\mathbf{T}$ from N image points (x_i, y_i), projections of N suitably chosen world points (X_i, Y_i, Z_i), using the system of equations obtained from the projection matrix. It turns out that if $N \geq 7$ and the points are not coplanar, the rank of the system is 7, and the system has a nontrivial solution (up to a scalar) that can be found using SVD methods (see Chapter 13, Trucco and Verri [27], or Jain, Katsuri, and Schunck [18]).

Another method consists in estimating the whole projection matrix M, which depends on 11 parameters, and then extracting extrinsic and intrinsic parameters. Again, SVD methods are used (see Trucco and Verri [27], and Faugeras [10]).

Cayley's formula can also be used to solve the calibration cameras, as explained in Faugeras [10]. Other problems in computer vision can be reduced to problems in projective geometry (see Faugeras [10]).

In computer graphics, it is also necessary to convert the 3D world coordinates of a point to a two-dimensional representation on a *view plane*. This is achieved by a so-called *viewing system* using a projective transformation. For details on viewing systems see Watt [31] or Foley, van Dam, Feiner, and Hughes [13].

Projective spaces are also the right framework to deal with rational curves and rational surfaces. Indeed, in the projective framework it is easy to deal with vanishing denominators and with “infinite” values of the parameter(s). Such an approach is presented in Chapter 22 for rational curves, and in Chapter 23 and 24 for rational surfaces. In fact, working in a projective framework yields a very simple proof of the method for drawing a rational curve as two Bézier segments (and similarly for surfaces).

It is much less obvious that projective geometry has applications to efficient communication, error-correcting codes, and cryptography, as very nicely explained by Beutelspacher and Rosenbaum [2]. We sketch these applications very briefly, referring our readers to [2] for details. We begin with efficient communication. Suppose that eight students would like to exchange information to do their homework economically. The idea is that each student solves part of the exercises and copies the rest from the others (which we do not recommend, of course!). It is assumed that each student solves his part of the homework at home, and that the solutions are communicated by phone. The problem is to minimize the number of phone calls. An obvious but expensive method is for each student to call each of the other seven students. A much better method is to imagine that the eight students are the vertices of a cube, say with coordinates from $\{0,1\}^3$. There are three types of edges:

1. Those parallel to the z-axis, called *type* 1;
2. Those parallel to the y-axis, called *type* 2;
3. Those parallel to the x-axis, called *type* 3.

The communication can proceed in three rounds as follows: All nodes connected by type 1 edges exchange solutions; all nodes connected by type 2 edges exchange solutions; and finally all nodes connected by type 3 edges exchange solutions.

It is easy to see that everybody has all the answers at the end of the three rounds. Furthermore, each student is involved only in three calls (making a call or receiving it), and the total number of calls is twelve.

In the general case, N nodes would like to exchange information in such a way that eventually every node has all the information. A good way to to this is to construct certain finite projective spaces, as explained in Beutelspacher and Rosenbaum [2]. We pick q to be an integer (for instance, a prime number) such that there is a finite projective space of any dimension over the finite field of order q. Then, we pick d such that

$$q^{d-1} < N \leq q^d.$$

Since q is prime, there is a projective space $\mathbf{P}(K^{d+1})$ of dimension d over the finite field K of order q, and letting $\mathscr{H}$ be the hyperplane at infinity in $\mathbf{P}(K^{d+1})$, we pick a frame $P_1,\ldots,P_d$ in $\mathscr{H}$. It turns out that the affine space $\mathscr{A} = \mathbf{P}(K^{d+1}) - \mathscr{H}$ has q^d points. Then the communication nodes can be identified with points in the affine space $\mathscr{A}$. Assuming for simplicity that $N = q^d$, the algorithm proceeds in d rounds.

During round i, each node $Q \in \mathscr{A}$ sends the information it has received to all nodes in $\mathscr{A}$ on the line QP_i.

It can be shown that at the end of the d rounds, each node has the total information, and that the total number of transactions is at most

$$(q-1)\log_q(N)N.$$

Other applications of projective spaces to communication systems with switches are described in Chapter 2, Section 8, of Beutelspacher and Rosenbaum [2]. Applications to error-correcting codes are described in Chapter 5 of the same book. Introducing even the most elementary notions of coding theory would take too much space. Let us simply say that the existence of certain types of good codes called *linear $[n, n-r]$-codes with minimum distance d* is equivalent to the existence of certain sets of points called $(n, d-1)$-sets in the finite projective space $\mathbf{P}(\{0,1\}^r)$. For the sake of completeness, a set of n points in a projective space is an *(n, s)-set* if s is the largest integer such that every subset of s points is projectively independent. For example, an $(n, 3)$-set is a set of n points no three of which are collinear, but at least four of them are coplanar.

Other applications of projective geometry to cryptography are given in Chapter 6 of Beutelspacher and Rosenbaum [2].

5.14 Problems

5.1. (a) Prove that for any field K and any $n \geq 0$, there is a bijection between $\mathbf{P}(K^{n+1})$ and $K^n \cup \mathbf{P}(K^n)$ (which allows us to identify them).

(b) For $K = \mathbb{R}$ or $\mathbb{C}$, prove that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are connected and compact.
Hint. Recall that $\mathbb{RP}^n = p(\mathbb{R}^{n+1})$ and $\mathbb{CP}^n = p(\mathbb{C}^{n+1})$. If

$$S^n = \{(x_1, \dots, x_{n+1}) \in K^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

prove that $p(S^n) = p(K^{n+1}) = \mathbf{P}(K^{n+1})$, and recall that S^n is compact for all $n \geq 0$ and connected for $n \geq 1$. For $n = 0$, $\mathbf{P}(K)$ consists of a single point.

5.2. Recall that $\mathbb{R}^2$ and $\mathbb{C}$ can be identified using the bijection $(x, y) \mapsto x + \mathrm{i}y$. Also recall that the subset $U(1) \subseteq \mathbb{C}$ consisting of all complex numbers of the form $\cos\theta + \mathrm{i}\sin\theta$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. If $c\colon U(1) \to U(1)$ is the map defined such that

$$c(z) = z^2,$$

prove that $c(z_1) = c(z_2)$ iff either $z_2 = z_1$ or $z_2 = -z_1$, and thus that c induces a bijective map $\widehat{c}\colon \mathbb{RP}^1 \to S^1$. Prove that $\widehat{c}$ is a homeomorphism (remember that $\mathbb{RP}^1$ is compact).

5.3. (i) In $\mathbb{R}^3$, the sphere S^2 is the set of points of coordinates (x,y,z) such that $x^2+y^2+z^2=1$. The point $N=(0,0,1)$ is called the *north pole*, and the point $S=(0,0,-1)$ is called the *south pole*. The *stereographic projection map* $\sigma_N\colon (S^2-\{N\})\to\mathbb{R}^2$ is defined as follows: For every point $M\neq N$ on S^2, the point $\sigma_N(M)$ is the intersection of the line through N and M and the plane of equation $z=0$. Show that if M has coordinates (x,y,z) (with $x^2+y^2+z^2=1$), then

$$\sigma_N(M)=\left(\frac{x}{1-z},\frac{y}{1-z}\right).$$

Prove that σ_N is bijective and that its inverse is given by the map $\tau_N\colon \mathbb{R}^2\to(S^2-\{N\})$, with

$$(x,y)\mapsto\left(\frac{2x}{x^2+y^2+1},\frac{2y}{x^2+y^2+1},\frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

Similarly, $\sigma_S\colon (S^2-\{S\})\to\mathbb{R}^2$ is defined as follows: For every point $M\neq S$ on S^2, the point $\sigma_S(M)$ is the intersection of the line through S and M and the plane of equation $z=0$. Show that

$$\sigma_S(M)=\left(\frac{x}{1+z},\frac{y}{1+z}\right).$$

Prove that σ_S is bijective and that its inverse is given by the map $\tau_S\colon \mathbb{R}^2\to(S^2-\{S\})$, with

$$(x,y)\mapsto\left(\frac{2x}{x^2+y^2+1},\frac{2y}{x^2+y^2+1},\frac{1-x^2-y^2}{x^2+y^2+1}\right).$$

Using the complex number $u=x+\mathrm{i}y$ to represent the point (x,y), the maps $\tau_N\colon\mathbb{R}^2\to(S^2-\{N\})$ and $\sigma_N\colon(S^2-\{N\})\to\mathbb{R}^2$ can be viewed as maps from $\mathbb{C}$ to $(S^2-\{N\})$ and from $(S^2-\{N\})$ to $\mathbb{C}$, defined such that

$$\tau_N(u)=\left(\frac{2u}{|u|^2+1},\frac{|u|^2-1}{|u|^2+1}\right)$$

and

$$\sigma_N(u,z)=\frac{u}{1-z},$$

and similarly for τ_S and σ_S. Prove that if we pick two suitable orientations for the xy-plane, we have

$$\sigma_N(M)\sigma_S(M)=1,$$

for every $M\in S^2-\{N,S\}$.

(ii) Identifying $\mathbb{C}^2$ and $\mathbb{R}^4$, for $z=x+\mathrm{i}y$ and $z'=x'+\mathrm{i}y'$, we define

$$\|(z,z')\|=\sqrt{x^2+y^2+x'^2+y'^2}.$$

The sphere S^3 is the subset of $\mathbb{C}^2$ (or $\mathbb{R}^4$) consisting of those points (z,z') such that $\|(z,z')\|^2 = 1$.

Prove that $\mathbf{P}(\mathbb{C}^2) = p(S^3)$, where $p\colon (\mathbb{C}^2 - \{(0,0)\}) \to \mathbf{P}(\mathbb{C}^2)$ is the projection map. If we let $u = z/z'$ (where $z,z' \in \mathbb{C}$) in the map

$$u \mapsto \left(\frac{2u}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right)$$

and require that $\|(z,z')\|^2 = 1$, show that we get the map $HF\colon S^3 \to S^2$ defined such that

$$HF((z,z')) = (2z\overline{z'}, |z|^2 - |z'|^2).$$

Prove that $HF\colon S^3 \to S^2$ induces a bijection $\widehat{HF}\colon \mathbf{P}(\mathbb{C}^2) \to S^2$, and thus that $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$ is homeomorphic to S^2.

(iii) Prove that the inverse image $HF^{-1}(s)$ of every point $s \in S^2$ is a circle. Thus S^3 can be viewed as a union of disjoint circles. The map HF is called the *Hopf fibration*.

5.4. (i) Prove that the *Veronese map* $V_2\colon \mathbb{R}^3 \to \mathbb{R}^6$ defined such that

$$V_2(x,y,z) = (x^2, y^2, z^2, yz, zx, xy)$$

induces a homeomorphism of $\mathbb{RP}^2$ onto $V_2(S^2)$. Show that $V_2(S^2)$ is a subset of the hyperplane $x_1 + x_2 + x_3 = 1$ in $\mathbb{R}^6$, and thus that $\mathbb{RP}^2$ is homeomorphic to a subset of $\mathbb{R}^5$. Prove that this homeomorphism is smooth.

(ii) Prove that the *Veronese map* $V_3\colon \mathbb{R}^4 \to \mathbb{R}^{10}$ defined such that

$$V_3(x,y,z,t) = (x^2, y^2, z^2, t^2, xy, yz, xz, xt, yt, zt)$$

induces a homeomorphism of $\mathbb{RP}^3$ onto $V_3(S^3)$. Show that $V_3(S^3)$ is a subset of the hyperplane $x_1 + x_2 + x_3 + x_4 = 1$ in $\mathbb{R}^{10}$, and thus that $\mathbb{RP}^3$ is homeomorphic to a subset of $\mathbb{R}^9$. Prove that this homeomorphism is smooth.

5.5. (i) Given a projective plane $\mathbf{P}(E)$ (over any field K) and any projective frame (a,b,c,d) in $\mathbf{P}(E)$, recall that a line is defined by an equation of the form $ux + vy + wz = 0$, where u,v,w are not all zero, and that two lines $ux + vy + wz = 0$ and $u'x + v'y + w'z = 0$ are identical iff $u' = \lambda u$, $v' = \lambda v$, and $w' = \lambda w$, for some $\lambda \neq 0$. Show that any two distinct lines $ux + vy + wz = 0$ and $u'x + v'y + w'z = 0$ intersect in a unique point of homogeneous coordinates

$$(vw' - wv', wu' - uw', uv' - vu').$$

(ii) Given a projective frame (a,b,c,d), let a' be the intersection of $\langle d,a\rangle$ and $\langle b,c\rangle$, b' be the intersection of $\langle d,b\rangle$ and $\langle a,c\rangle$, and c' be the intersection of $\langle d,c\rangle$ and $\langle a,b\rangle$. Show that the points a',b',c' have homogeneous coordinates $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$. Let e be the intersection of $\langle b,c\rangle$ and $\langle b',c'\rangle$, f be the intersection of $\langle a,c\rangle$ and $\langle a',c'\rangle$, and g be the intersection of $\langle a,b\rangle$ and $\langle a',b'\rangle$. Show that

e, f, g have homogeneous coordinates $(0,-1,1)$, $(1,0,-1)$, and $(-1,1,0)$, and thus that the points e, f, g are on the line of equation $x+y+z=0$.

5.6. Prove that if $(a_i)_{1\leq i\leq n+2}$ is a projective frame, then each subfamily $(a_j)_{j\neq i}$ is projectively independent.

5.7. (i) Given a projective space $\mathbf{P}(E)$ of dimension 3 (over any field K) and any projective frame (A,B,C,D,E) in $\mathbf{P}(E)$, recall that a plane is defined by an equation of the form $ux_0+vx_1+wx_2+tx_3=0$ where u,v,w,t are not all zero.

Letting (a_0,a_1,a_2,a_3), (b_0,b_1,b_2,b_3), (c_0,c_1,c_2,c_3), and (d_0,d_1,d_2,d_3) be the homogeneous coordinates of some points a,b,c,d with respect to the projective frame (A,B,C,D,E), prove that a,b,c,d are coplanar iff

$$\begin{vmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

(ii) Two tetrahedra (A,B,C,D) and (A',B',C',D') are called *Möbius tetrahedra* if A,B,C,D belong respectively to the planes $\langle B',C',D'\rangle$, $\langle C',D',A'\rangle$, $\langle D',A',B'\rangle$, and $\langle A',B',C'\rangle$, and also if A',B',C',D' belong respectively to the planes $\langle B,C,D\rangle$, $\langle C,D,A\rangle$, $\langle D,A,B\rangle$, and $\langle A,B,C\rangle$.

Prove that if A,B,C,D belong respectively to the planes $\langle B',C',D'\rangle$, $\langle C',D',A'\rangle$, $\langle D',A',B'\rangle$, and $\langle A',B',C'\rangle$, and if A',B',C' belong respectively to the planes $\langle B,C,D\rangle$, $\langle C,D,A\rangle$, and $\langle D,A,B\rangle$, then D' belongs to $\langle A,B,C\rangle$. Prove that Möbius tetrahedra exist (Möbius, 1828).

Hint. Let (A,B,C,D,E) be a projective frame based on A,B,C,D. Find the conditions expressing that A',B',C',D' belong respectively to the planes $\langle B,C,D\rangle$, $\langle C,D,A\rangle$, $\langle D,A,B\rangle$, and $\langle A,B,C\rangle$, that A',B',C',D' are not coplanar, and that A,B,C,D belong respectively to the planes $\langle B',C',D'\rangle$, $\langle C',D',A'\rangle$, $\langle D',A',B'\rangle$, and $\langle A',B',C'\rangle$. Show that these conditions are compatible.

5.8. Show that if we relax the hypotheses of Lemma 5.5 to $(a_i)_{1\leq i\leq n+2}$ being a projective frame in $\mathbf{P}(E)$ and $(b_i)_{1\leq i\leq n+2}$ being any $n+2$ points in $\mathbf{P}(F)$, then there may be no projective map $h\colon \mathbf{P}(E)\to\mathbf{P}(F)$ such that $h(a_i)=b_i$ for $1\leq i\leq n+2$, or h may not be necessarily unique or bijective.

5.9. For every i, $1\leq i\leq n+1$, let U_i be the subset of $\mathbb{RP}^n=\mathbf{P}(\mathbb{R}^{n+1})$ consisting of all points of homogeneous coordinates $(x_1,\ldots,x_i,\ldots,x_{n+1})$ such that $x_i\neq 0$. Show that U_i is an open subset of $\mathbb{RP}^n$. Show that $U_i\cap U_j\neq\emptyset$ for all i,j. Show that there is a bijection between U_i and $\mathbb{A}^n$ defined such that

$$(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_{n+1})\mapsto\left(\frac{x_1}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_{n+1}}{x_i}\right),$$

whose inverse is the map

$$(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_{i-1},1,x_i,\ldots,x_n).$$

Does the above result extend to $\mathbb{P}^n_K$ where K is any field?

5.10. (i) Given an affine space $\left(E, \overrightarrow{E}\right)$ (over any field K), prove that there is a bijection between affine subspaces of E and projective subspaces of $\widetilde{E}$ not contained in $\mathbf{P}(\overrightarrow{E})$.

(ii) Prove that two affine subspaces of E are parallel iff the corresponding projective subspaces of $\widetilde{E}$ have the same intersection with the hyperplane at infinity $\mathbf{P}(\overrightarrow{E})$.

(iii) Prove that there is a bijection between affine maps from E to F and projective maps from $\widetilde{E}$ to $\widetilde{F}$ mapping the hyperplane at infinity $\mathbf{P}(\overrightarrow{E})$ into the hyperplane at infinity $\mathbf{P}(\overrightarrow{F})$.

5.11. (i) Consider the map $\varphi\colon \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3$ defined such that

$$\varphi((x_0,x_1), (y_0,y_1)) = (x_0y_0, x_0y_1, x_1y_0, x_1y_1),$$

where (x_0,x_1) and (y_0,y_1) are homogeneous coordinates on $\mathbb{RP}^1$. Prove that φ is well-defined and that $\varphi(\mathbb{RP}^1 \times \mathbb{RP}^1)$ is equal the algebraic subset of $\mathbb{RP}^3$ defined by the homogeneous equation

$$w_{0,0}\, w_{1,1} = w_{0,1}\, w_{1,0},$$

where $(w_{0,0}, w_{0,1}, w_{1,0}, w_{1,1})$ are homogeneous coordinates on $\mathbb{RP}^3$.
Hint. Show that if $w_{0,0}\, w_{1,1} = w_{0,1}\, w_{1,0}$ and for instance $w_{0,0} \neq 0$, then

$$\varphi((w_{0,0},\, w_{1,0}),\, (w_{0,0},\, w_{0,1})) = w_{0,0}(w_{0,0},\, w_{0,1},\, w_{1,0},\, w_{1,1}),$$

and since $w_{0,0}(w_{0,0},\, w_{0,1},\, w_{1,0},\, w_{1,1})$ and $(w_{0,0},\, w_{0,1},\, w_{1,0},\, w_{1,1})$ are equivalent homogeneous coordinates, the result follows.

Prove that φ is injective.

For $x = (x_0,x_1) \in \mathbb{RP}^1$, show that $\varphi(\{x\} \times \mathbb{RP}^1)$ is a line L^1_x in $\mathbb{RP}^3$, that $L^1_x \cap L^1_{x'} = \emptyset$ whenever $L^1_x \neq L^1_{x'}$, and that the union of all these lines is equal to $\varphi(\mathbb{RP}^1 \times \mathbb{RP}^1)$. Similarly, for $y = (y_0,y_1) \in \mathbb{RP}^1$, show that $\varphi(\mathbb{RP}^1 \times \{y\})$ is a line L^2_y in $\mathbb{RP}^3$, that $L^2_y \cap L^2_{y'} = \emptyset$ whenever $L^2_y \neq L^2_{y'}$, and that the union of all these lines is equal to $\varphi(\mathbb{RP}^1 \times \mathbb{RP}^1)$. Also prove that $L^1_x \cap L^2_y$ consists of a single point.

The embedding φ is called the *Segre embedding*. It shows that $\mathbb{RP}^1 \times \mathbb{RP}^1$ can be embedded as a quadric surface in $\mathbb{RP}^3$. Do the above results extend to $\mathbb{P}^1_K \times \mathbb{P}^1_K$ and $\mathbb{P}^3_K$ where K is any field? Draw as well as possible the affine part of $\varphi(\mathbb{RP}^1 \times \mathbb{RP}^1)$ in $\mathbb{R}^3$ corresponding to $w_{1,1} = 1$.

(ii) Consider the map $\varphi\colon \mathbb{RP}^m \times \mathbb{RP}^n \to \mathbb{RP}^N$ where $N = (m+1)(n+1) - 1$, defined such that

$$\varphi((x_0,\ldots,x_m),\, (y_0,\ldots,y_n)) = (x_0y_0,\, \ldots, x_0y_n,\, x_1y_0,\, \ldots, x_1y_n, \ldots, x_my_0,\, \ldots, x_my_n),$$

where $(x_0, \ldots, x_m)$ and $(y_0, \ldots, y_n)$ are homogeneous coordinates on $\mathbb{RP}^m$ and $\mathbb{RP}^n$. Prove that φ is well-defined and that $\varphi(\mathbb{RP}^m \times \mathbb{RP}^n)$ is equal the algebraic subset of $\mathbb{RP}^N$ defined by the set of homogeneous equations

$$\begin{vmatrix} w_{i,j} & w_{i,l} \\ w_{k,j} & w_{k,l} \end{vmatrix} = 0,$$

where $0 \leq i, k \leq m$ and $0 \leq j, l \leq n$, and where $(w_{0,0}, \ldots, w_{0,m}, \ldots, w_{m,0}, \ \ldots, w_{m,n})$ are homogeneous coordinates on $\mathbb{RP}^N$.
Hint. Show that if

$$\begin{vmatrix} w_{i,j} & w_{i,l} \\ w_{k,j} & w_{k,l} \end{vmatrix} = 0,$$

where $0 \leq i, k \leq m$ and $0 \leq j, l \leq n$ and for instance $w_{0,0} \neq 0$, then

$$\varphi(x, y) = w_{0,0}(w_{0,0}, \ldots, w_{0,m}, \ldots, w_{m,0}, \ldots, w_{m,n}),$$

where $x = (w_{0,0}, \ldots, w_{m,0})$ and $y = (w_{0,0}, \ldots, w_{0,n})$.

Prove that φ is injective. The embedding φ is also called the *Segre embedding*. It shows that $\mathbb{RP}^m \times \mathbb{RP}^n$ can be embedded as an algebraic variety in $\mathbb{RP}^N$. Do the above results extend to $\mathbb{P}^m_K \times \mathbb{P}^n_K$ and $\mathbb{P}^N_K$ where K is any field?

5.12. (i) In the projective space $\mathbb{RP}^3$, a line D is determined by two distinct hyperplanes of equations

$$\begin{aligned} \alpha x + \beta y + \gamma z + \delta t &= 0, \\ \alpha' x + \beta' y + \gamma' z + \delta' t &= 0, \end{aligned}$$

where $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ are linearly independent.

Prove that the equations of the two hyperplanes defining D can always be written either as

$$\begin{aligned} x_1 &= a x_3 + a' x_4, \\ x_2 &= b x_3 + b' x_4, \end{aligned}$$

where $\{x_1, x_2, x_3, x_4\} = \{x, y, z, t\}$, $\{x_1, x_2\} \subseteq \{x, y, z\}$, and either $a \neq 0$ or $b \neq 0$, or as

$$\begin{aligned} t &= 0, \\ lx + my + nz &= 0, \end{aligned}$$

where $l \neq 0$, $m \neq 0$, or $n \neq 0$.

In the first case, prove that D is also determined by the intersection of three hyperplanes whose equations are of the form

$$\begin{aligned} cy - bz &= lt, \\ az - cx &= mt, \end{aligned}$$

$$bx - ay = nt,$$

where the equation

$$al + bm + cn = 0$$

holds, and where $a \neq 0$, $b \neq 0$, or $c \neq 0$. We can view (a, b, c, l, m, n) as homogeneous coordinates in $\mathbb{RP}^5$ associated with D. In the case where the equations of D are

$$\begin{aligned} t &= 0, \\ lx + my + nz &= 0, \end{aligned}$$

we let $(0, 0, 0, l, m, n)$ be the homogeneous coordinates associated with D. Of course, $al + bm + cn = 0$ holds. The homogeneous coordinates (a, b, c, l, m, n) such that $al + bm + cn = 0$ are called the *Plücker coordinates* of D.

(ii) Conversely, given some homogeneous coordinates (a, b, c, l, m, n) in $\mathbb{RP}^5$ satisfying the equation

$$al + bm + cn = 0,$$

show that there is a unique line D with Plücker coordinates (a, b, c, l, m, n).
Hint. If $a = b = c = 0$, the corresponding line has equations

$$\begin{aligned} t &= 0, \\ lx + my + nz &= 0. \end{aligned}$$

Otherwise, the equations

$$\begin{aligned} cy - bz &= lt, \\ az - cx &= mt, \\ bx - ay &= nt, \end{aligned}$$

are compatible, and they determine a unique line D with Plücker coordinates (a, b, c, l, m, n).

Conclude that the lines in $\mathbb{RP}^3$ can be viewed as the algebraic subset of $\mathbb{RP}^5$ defined by the homogeneous equation

$$x_1 x_3 + x_2 x_5 + x_3 x_6 = 0.$$

This quadric surface in $\mathbb{RP}^5$ is an example of a *Grassmannian variety*. It is often called the *Klein quadric*. Do the above results extend to lines in $\mathbb{P}^3_K$ and $\mathbb{P}^5_K$ where K is any field?

5.13. Given any two distinct point $a, b \in \mathbb{RP}^3$ of homogeneous coordinates (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4), let p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23} be the numbers defined as follows:

$$p_{12} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad p_{13} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \qquad p_{14} = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix},$$
$$p_{34} = \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}, \qquad p_{42} = \begin{vmatrix} a_4 & a_2 \\ b_4 & b_2 \end{vmatrix}, \qquad p_{23} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

(i) Prove that

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

Hint. Expand the determinant

$$\begin{vmatrix} a_1 & b_1 & a_1 & b_1 \\ a_2 & b_2 & a_2 & b_2 \\ a_3 & b_3 & a_3 & b_3 \\ a_4 & b_4 & a_4 & b_4. \end{vmatrix}$$

Conversely, given any six numbers satisfying the equation

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,$$

prove that two points $a = (a_1, a_2, a_3, 0)$ and $b = (b_1, 0, b_3, b_4)$ can be determined such that the p_{ij} are associated with a and b.
Hint. Show that the equations

$$\begin{aligned} -a_2b_1 &= p_{12}, \\ a_3b_4 &= p_{34}, \\ a_1b_3 - a_3b_1 &= p_{13}, \\ -a_2b_4 &= p_{42}, \\ a_1b_4 &= p_{14}, \\ a_2b_3 &= p_{23}, \end{aligned}$$

are solvable iff

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

The tuple $(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$ can be viewed as homogeneous coordinates in $\mathbb{RP}^5$ of the line $\langle a, b\rangle$. They are the *Plücker coordinates* of $\langle a, b\rangle$.

(ii) Prove that two lines of Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$ and $(p'_{12}, p'_{13}, p'_{14}, p'_{34}, p'_{42}, p'_{23})$ intersect iff

$$p_{12}p'_{34} + p_{13}p'_{42} + p_{14}p'_{23} + p_{34}p'_{12} + p_{42}p'_{13} + p_{23}p'_{14} = 0.$$

Thus, the set of lines that meet a given line in $\mathbb{RP}^3$ correspond to a set of points in $\mathbb{RP}^5$ belonging to a hyperplane, as well as to the Klein quadric. Do the above results extend to lines in $\mathbb{P}^3_K$ and $\mathbb{P}^5_K$ where K is any field?

(iii) Three lines L_1, L_2, L_3 in $\mathbb{RP}^3$ are mutually skew lines iff no pairs of any two of these lines are coplanar. Given any three mutually skew lines L_1, L_2, L_3 and any four lines M_1, M_2, M_3, M_4 in $\mathbb{RP}^3$ such that each line M_i meets every line L_j, show that if any line L meets three of the four lines M_1, M_2, M_3, M_4, then it also meets the fourth. Does the above result extend to $\mathbb{P}^3_K$ where K is any field? Show that the set of lines meeting three given mutually skew lines L_1, L_2, L_3 in $\mathbb{P}^3_K$ is a ruled quadric surface. What do the affine pieces of this quadric look like in $\mathbb{R}^3$?

(iv) Four lines L_1, L_2, L_3, L_4 in $\mathbb{RP}^3$ are mutually skew lines iff no pairs of any two of these lines are coplanar. Given any four mutually skew lines L_1, L_2, L_3, L_4, show that there are at most two lines meeting all four of them. In $\mathbb{CP}^3$, show that there are either two distinct lines or a double line meeting all four of them.

5.14. (i) Prove that the cross-ratio $[a,b,c,d]$ is invariant if any two elements and the complementary two elements are transposed. Prove that

$$[a,b,c,d] = [b,a,c,d]^{-1} = [a,b,d,c]^{-1}$$

and that

$$[a,b,c,d] = 1 - [a,c,b,d].$$

(ii) Letting $\lambda = [a,b,c,d]$, prove that if $\lambda \in \{\infty, 0, 1\}$, then any permutation of $\{a,b,c,d\}$ yields a cross-ratio in $\{\infty, 0, 1\}$, and if $\lambda \notin \{\infty, 0, 1\}$, then there are at most the six values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad 1-\frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1}.$$

(iii) Prove that the function

$$\lambda \mapsto \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$$

takes a constant value on the six values listed in part (ii).

5.15. Viewing a point (x,y) in $\mathbb{A}^2$ as the complex number $z = x+iy$, prove that four points (a,b,c,d) are cocyclic or collinear iff the cross-ratio $[a,b,c,d]$ is a real number.

5.16. Given any distinct points (x_1,x_2,x_3,x_4) in $\mathbb{RP}^1$, prove that they form a harmonic division, i.e., $[x_1,x_2,x_3,x_4] = -1$ iff

$$2(x_1x_2 + x_3x_4) = (x_1+x_2)(x_3+x_4).$$

Prove that $[0,x_2,x_3,x_4] = -1$ iff

$$\frac{2}{x_2} = \frac{1}{x_3} + \frac{1}{x_4}.$$

Prove that $[x_1,x_2,x_3,\infty] = -1$ iff

$$2x_3 = x_1 + x_2.$$

Do the above results extend to $\mathbb{P}^1_K$ where K is any field?

5.17. Consider the quadrangle (projective frame) (a,b,c,d) in a projective plane, and let a' be the intersection of $\langle d,a\rangle$ and $\langle b,c\rangle$, b' be the intersection of $\langle d,b\rangle$ and $\langle a,c\rangle$, and c' be the intersection of $\langle d,c\rangle$ and $\langle a,b\rangle$. Show that the following quadruples of lines form harmonic divisions: $(\langle c,a\rangle,\langle b',a'\rangle,\langle d,b\rangle,\langle b',c'\rangle)$, $(\langle b,a\rangle,\langle c',a'\rangle,\langle d,c\rangle,\langle c',b'\rangle)$, and $(\langle b,c\rangle, \langle a',b'\rangle, \langle a,d\rangle, \langle a',c'\rangle)$.
Hint. Send some suitable lines to infinity.

5.18. Let $\mathbf{P}(E)$ be a projective space over any field. For any projective map $\mathbf{P}(f)\colon \mathbf{P}(E)\to\mathbf{P}(E)$, a point $a=p(u)$ is a fixed point of $\mathbf{P}(f)$ iff $\mathbf{P}(f)(a)=a$. Prove that $a=p(u)$ is a fixed point of $\mathbf{P}(f)$ iff u is an eigenvector of the linear map $f\colon E\to E$. Prove that if $E=\mathbb{R}^{2n+1}$, then every projective map $\mathbf{P}(f)\colon \mathbb{RP}^{2n}\to\mathbb{RP}^{2n}$ has a fixed point. Prove that if $E=\mathbb{C}^{n+1}$, then every projective map $\mathbf{P}(f)\colon \mathbb{CP}^{n}\to\mathbb{CP}^{n}$ has a fixed point.

5.19. A projectivity $\mathbf{P}(f)\colon \mathbb{RP}^{n}\to\mathbb{RP}^{n}$ is an *involution* if $\mathbf{P}(f)$ is not the identity and if $\mathbf{P}(f)\circ\mathbf{P}(f)=\mathrm{id}$. Prove that a projectivity $\mathbf{P}(f)\colon \mathbb{RP}^{1}\to\mathbb{RP}^{1}$ is an involution iff the trace of the matrix of f is null. Does the above result extend to $\mathbb{P}^1_K$ where K is any field?

5.20. Recall Desargues's theorem in the plane: Given any two triangles (a,b,c) and (a',b',c') in $\mathbb{RP}^2$, where the points a,b,c,a',b',c' are distinct and the lines $A=\langle b,c\rangle$, $B=\langle a,c\rangle$, $C=\langle a,b\rangle$, $A'=\langle b',c'\rangle$, $B'=\langle a',c'\rangle$, $C'=\langle a',b'\rangle$ are distinct, if the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, and $\langle c,c'\rangle$ intersect in a common point d distinct from a,b,c,a',b',c', then the intersection points $p=\langle b,c\rangle\cap\langle b',c'\rangle$, $q=\langle a,c\rangle\cap\langle a',c'\rangle$, and $r=\langle a,b\rangle\cap\langle a',b'\rangle$ belong to a common line distinct from A,B,C,A',B',C'.

Prove that the dual of the above result is its converse. Deduce Desargues's theorem: Given any two triangles (a,b,c) and (a',b',c') in $\mathbb{RP}^2$, where the points a,b,c,a',b',c' are distinct and the lines $A=\langle b,c\rangle, B=\langle a,c\rangle, C=\langle a,b\rangle, A'=\langle b',c'\rangle$, $B'=\langle a',c'\rangle$, $C'=\langle a',b'\rangle$ are distinct, the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, and $\langle c,c'\rangle$ intersect in a common point d distinct from a,b,c,a',b',c' iff the intersection points $p=\langle b,c\rangle\cap\langle b',c'\rangle$, $q=\langle a,c\rangle\cap\langle a',c'\rangle$, and $r=\langle a,b\rangle\cap\langle a',b'\rangle$ belong to a common line distinct from A,B,C,A',B',C'.

Do the above results extend to $\mathbb{P}^2_K$ where K is any field?

5.21. Let D and D' be any two distinct lines in the real projective plane $\mathbb{RP}^2$, and let $f\colon D\to D'$ be a projectivity. Prove the following facts.

(1) If f is a perspectivity, then for any two distinct points m,n on D, the lines $\langle m,f(n)\rangle$ and $\langle n,f(m)\rangle$ intersect on some fixed line passing through $D\cap D'$.
Hint. Consider any three distinct points a,b,c on D and use Desargues's theorem.

(2) If f is not a perspectivity, then for any two distinct points m,n on D, the lines $\langle m,f(n)\rangle$ and $\langle n,f(m)\rangle$ intersect on the line passing through $f(D\cap D')$ and $f^{-1}(D\cap D')$.
Hint. Use some suitable composition of perspectivities. The line passing through $f(D\cap D')$ and $f^{-1}(D\cap D')$ is called the *axis* of the projectivity.

(iii) Prove that any projectivity $f\colon D\to D'$ between distinct lines is the composition of two perspectivities.

(iv) Use the above facts to give a quick proof of Pappus's theorem: Given any two distinct lines D and D' in a projective plane, for any distinct points a,b,c,a',b',c' with a,b,c on D and a',b',c' on D', if a,b,c,a',b',c' are distinct from the intersection of D and D', then the intersection points $p = \langle b,c'\rangle \cap \langle b',c\rangle$, $q = \langle a,c'\rangle \cap \langle a',c\rangle$, and $r = \langle a,b'\rangle \cap \langle a',b\rangle$ are collinear.

Do the above results extend to $\mathbb{P}^2_K$ where K is any field?

5.22. Recall that in the real projective plane $\mathbb{RP}^2$, by duality, a point a corresponds to the pencil of lines a^* passing through a.

(i) Given any two distinct points a and b in the real projective plane $\mathbb{RP}^2$ and any line L containing neither a nor b, the *perspectivity of axis L between a^* and b^** is the map $f\colon a^* \to b^*$ defined such that for every line $D \in a^*$, the line $f(D)$ is the line through b and the intersection of D and L.

Prove that a projectivity $f\colon a^* \to b^*$ is a perspectivity iff $f(\langle a,b\rangle) = \langle b,a\rangle$.

(ii) Prove that a bijection $f\colon a^* \to b^*$ is a projectivity iff it preserves the cross-ratios of any four distinct lines in the pencil a^*.

(iii) State and prove the dual of Pappus's theorem.

Do the above results extend to $\mathbb{P}^2_K$ where K is any field?

5.23. (i) Prove that every projectivity $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ has at most 2 fixed points. A projectivity $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ is called *elliptic* if it has no fixed points, *parabolic* if it has a single fixed point, *hyperbolic* if it has two distinct fixed points. Prove that every projectivity $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ has 2 distinct fixed points or a double fixed point.

(ii) Recall that a projectivity $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ is an *involution* if f is not the identity and if $f \circ f = \mathrm{id}$. Prove that f is an involution iff there is some point $a \in \mathbb{RP}^1$ such that $f(a) \neq a$ and $f(f(a)) = a$.

(iii) Given any two distinct points $a,b \in \mathbb{RP}^1$, prove that there is a unique involution $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ having a and b as fixed points. Furthermore, for all $m \neq a,b$, we have

$$[a,b,m,f(m)] = -1.$$

Conversely, the above formula defines an involution with fixed points a and b.

(iv) Prove that every projectivity $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ is the composition of at most two involutions.

Do the above results extend to $\mathbb{P}^1_K$ where K is any field?

5.24. Prove that an involution $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ has zero or two distinct fixed points. Prove that an involution $f\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ has two distinct fixed points.

5.25. Prove that a bijection $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ having two distinct fixed points a and b is a projectivity iff there is some $k \neq 0$ in $\mathbb{R}$ such that for all $m \neq a,b$, we have

$$[a,b,m,f(m)] = k.$$

Does the above result extend to $\mathbb{P}^1_K$ where K is any field?

5.26. Prove that every projectivity $f\colon \mathbb{RP}^1 \to \mathbb{RP}^1$ is the composition of at most three perspectivities.

Hint. Consider some appropriate perspectivities.

Does the above result extend to $\mathbb{P}^1_K$ where K is any field?

5.27. Let (a,b,c,d) be a projective frame in $\mathbb{RP}^2$, and let D be a line not passing through any of a,b,c,d. The line D intersects $\langle a,b\rangle$ and $\langle c,d\rangle$ in p and p', $\langle b,c\rangle$ and $\langle a,d\rangle$ in q and q', and $\langle b,d\rangle$ and $\langle a,c\rangle$ in r and r'. Prove that there is a unique involution mapping p to p', q to q', and r to r'.
Hint. Consider some appropriate perspectivities.

Does the above result extend to $\mathbb{P}^2_K$ where K is any field?

5.28. Let (a,b,c) be a triangle in $\mathbb{RP}^2$, and let D be a line not passing through any of a,b,c, so that D intersects $\langle b,c\rangle$ in p, $\langle c,a\rangle$ in q, and $\langle a,b\rangle$ in r. Let L_a, L_b, L_c be three lines passing through a,b,c, respectively, and intersecting D in p', q', r'. Prove that there is a unique involution mapping p to p', q to q', and r to r' iff the lines L_a, L_b, L_c are concurrent.
Hint. Use Problem 5.27.

Does the above result extend to $\mathbb{P}^2_K$ where K is any field?

5.29. In a projective plane $\mathbf{P}(E)$ where E is a vector space of dimension 3 over any field K, a *conic* is the set of points of homogeneous coordinates (x,y,z) such that

$$\alpha x^2 + \beta y^2 + 2\gamma xy + 2\delta xz + 2\lambda yz + \mu z^2 = 0,$$

where $(\alpha,\beta,\gamma,\delta,\lambda,\mu) \neq (0,0,0,0,0,0)$. We can write the equation of the conic as

$$(x,y,z)\begin{pmatrix} \alpha & \gamma & \delta \\ \gamma & \beta & \lambda \\ \delta & \lambda & \mu \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

and letting

$$A = \begin{pmatrix} \alpha & \gamma & \delta \\ \gamma & \beta & \lambda \\ \delta & \lambda & \mu \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the equation of the conic becomes

$$X^\top A X = 0.$$

We say that a conic of equation $X^\top A X = 0$ is *nondegenerate* if $\det(A) \neq 0$ and *degenerate* if $\det(A) = 0$.

(i) For $K = \mathbb{R}$, show that there is only one type of nondegenerate conic, and that there are three kinds of degenerate conics: two distinct lines, a double line, a point, and the empty set. For $K = \mathbb{C}$, show that there is only one type of nondegenerate conic, and that there are two kinds of degenerate conics: two distinct lines or a double line.

(ii) Given any two distinct points a and b in $\mathbb{RP}^2$ and any projectivity $f\colon a^* \to b^*$ that is not a perspectivity, prove that the set of points of the form $L \cap f(L)$ is a nondegenerate conic, where L is any line in the pencil a^*.

What happens when f is a perspectivity? Does the above result hold for any field K?

(iii) Given a nondegenerate conic C, for any point $a \in C$ we can define a bijection $j_a\colon a^* \to C$ as follows: For every line L through a, we define $j_a(L)$ as the other intersection of L and C when L is not the tangent to C at a, and $j_a(L) = a$ otherwise. Given any two distinct points $a, b \in C$, show that the map $f = j_b^{-1} \circ j_a$ is a projectivity $f\colon a^* \to b^*$ that is not a perspectivity. In fact, if O is the intersection of the tangents to C at a and b, show that $f(\langle O, a\rangle) = \langle b, a\rangle$, $f(\langle a, b\rangle) = \langle b, O\rangle$, and for any point $m \neq a, b$ on C, $f(\langle a, m\rangle) = \langle b, m\rangle$. Conclude that C is the set of points of the form $L \cap f(L)$, where L is any line in the pencil a^*.
Hint. In a projective frame where $a = (1,0,0)$ and $b = (0,1,0)$, the equation of a conic is of the form

$$pz^2 + qxy + ryz + sxz = 0.$$

Remark: The above characterization of the conics is due to Steiner (and Chasles).

(iv) Prove that six points (a, b, c, d, e, f) such that no three of them are collinear belong to a conic iff

$$[\langle a, c\rangle, \langle a, d\rangle, \langle a, e\rangle, \langle a, f\rangle] = [\langle b, c\rangle, \langle b, d\rangle, \langle b, e\rangle, \langle b, f\rangle].$$

5.30. Given a nondegenerate conic C and any six points a, b, c, d, e, f on C such that no three of them are collinear, prove *Pascal's theorem*: The points $z = \langle a, b\rangle \cap \langle d, e\rangle$, $w = \langle b, c\rangle \cap \langle e, f\rangle$, and $t = \langle c, d\rangle \cap \langle f, a\rangle$ are collinear.

Recall that the line $\langle a, a\rangle$ is interpreted as the tangent to C at a.
Hint. By Problem 5.29, for any point m on the conic C, the bijection $j_m\colon m^* \to C$ allows the definition of the cross-ratio of four points a, b, c, d on C as the cross ratio of the lines $\langle m, a\rangle$, $\langle m, b\rangle$, $\langle m, c\rangle$, and $\langle m, d\rangle$ (which does not depend on m). Also recall that the cross-ratio of four lines in the pencil m^* is equal to the cross-ratio of the four intersection points with any line not passing through m. Prove that

$$[z, x, d, e] = [t, c, d, y],$$

and use the perspectivity of center w between $\langle c, y\rangle$ and $\langle e, x\rangle$.

5.31. In a projective plane $\mathbf{P}(E)$ where E is a vector space of dimension 3 over any field K of characteristic different from 2 (say, $K = \mathbb{R}$ or $K = \mathbb{C}$), given a conic C of equation $F(x, y, z) = 0$ where

$$F(x,y,z) = \alpha x^2 + \beta y^2 + 2\gamma xy + 2\delta xz + 2\lambda yz + \mu z^2 = 0$$

(with $(\alpha, \beta, \gamma, \delta, \lambda, \mu) \neq (0,0,0,0,0,0)$), using the notation of Problem 5.29 with $X^\top = (x, y, z)$ and $Y^\top = (u, v, w)$, verify that

$$Y^\top AX = \frac{1}{2}\,(uF'_x + vF'_y + wF'_z),$$

where F'_x, F'_y, F'_z denote the partial derivatives of $F(x,y,z)$.

If the conic C of equation $X^\top AX = 0$ is nondegenerate, it is well known (and easy to prove) that the tangent line to C at (x_0, y_0, z_0) is given by the equation

$$xF'_{x_0} + yF'_{y_0} + zF'_{z_0} = 0,$$

and thus by the equation $X^\top AX_0 = 0$, with $X^\top = (x,y,z)$ and $X_0^\top = (x_0, y_0, z_0)$. Therefore, the equation of the tangent to C at (x_0, y_0, z_0) is of the form

$$ux + vy + wz = 0,$$

where

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad \text{and} \quad (x_0, y_0, z_0)A \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0.$$

(i) If C is a nondegenerate conic of equation $X^\top AX = 0$ in the projective plane $\mathbf{P}(E)$, prove that the set C^* of tangent lines to C is a conic of equation $Y^\top A^{-1} Y = 0$ in the projective plane $\mathbf{P}(E^*)$, where E^* is the dual of the vector space E. Prove that $C^{**} = C$.

Remark: The conic C is sometimes called a *point conic* and the conic C^* a *line conic*. The set of lines defined by the conic C^* is said to be the *envelope* of the conic C.

Conclude that duality transforms the points of a nondegenerate conic into the tangents of the conic, and the tangents of the conic into the points of the conic.

(ii) Given any two distinct lines L and M in $\mathbb{RP}^2$ and any projectivity $f\colon L \to M$ that is not a perspectivity, prove that the lines of the form $\langle a, f(a)\rangle$ are the tangents enveloping a nondegenerate conic, where a is any point on the line L (use duality).

What happens when f is a perspectivity? Does the above result hold for any field K?

(iii) Given a nondegenerate conic C, for any two distinct tangents L and M to C at a and b, if $O = L \cap M$, show that the map $f\colon L \to M$ defined such that $f(a) = O$, $f(O) = b$, and $f(L \cap T) = M \cap T$ for any tangent $T \neq L, M$ is a projectivity. Conclude that C is the envelope of the set of lines of the form $\langle m, f(m)\rangle$, where m is any point on L (use duality).

5.32. Given a nondegenerate conic C, prove *Brianchon's theorem*: For any hexagon (a, b, c, d, e, f) circumscribed about C (which means that $\langle a,b\rangle$, $\langle b,c\rangle$, $\langle c,d\rangle$, $\langle d,e\rangle$, $\langle e,f\rangle$, and $\langle f,a\rangle$ are tangent to C), the diagonals $\langle a,d\rangle$, $\langle b,e\rangle$, and $\langle c,f\rangle$ are concurrent.
Hint. Use duality.

5.33. (a) Consider the map $\mathscr{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x,y,z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathscr{H}(x,y,z) = \mathscr{H}(x',y',z')$ iff $(x',y',z') = (x,y,z)$ or $(x',y',z') = (-x,-y,-z)$. In other words, the inverse image of every point in $\mathscr{H}(S^2)$ consists of two antipodal points.

Prove that the map $\mathscr{H}$ induces an injective map from the projective plane onto $\mathscr{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathscr{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ by choosing any parametrization of the sphere S^2 and applying the map $\mathscr{H}$ to it. Actually, it turns out to be more convenient to use the map $\mathscr{A}$ defined such that

$$(x,y,z) \mapsto (2xy, 2yz, 2xz, x^2 - y^2),$$

because it yields nicer parametrizations. For example, using the stereographic representation where

$$\begin{aligned} x(u,v) &= \frac{2u}{u^2+v^2+1}, \\ y(u,v) &= \frac{2v}{u^2+v^2+1}, \\ z(u,v) &= \frac{u^2+v^2-1}{u^2+v^2+1}, \end{aligned}$$

show that the following parametrization of the projective plane in $\mathbb{R}^4$ is obtained:

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ t(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

Investigate the surfaces in $\mathbb{R}^3$ obtained by dropping one of the four coordinates. Show that there are only two of them (up to a rigid motion).

5.34. Give the details of the proof that the altitudes of a triangle are concurrent.

5.35. Let K be the finite field $K = \{0,1\}$. Prove that the projective plane $\mathbf{P}(K^3)$ contains 7 points and 7 lines. Draw the configuration formed by these seven points and lines.

5.36. Prove that if P and Q are two homogeneous polynomials of degree 2 over $\mathbb{R}$ and if $V(P) = V(Q)$ contains at least three elements, then there is some $\lambda \in \mathbb{R}$ such that $Q = \lambda P$, with $\lambda \neq 0$.
Hint. Choose some convenient frame.

5.37. In the Euclidean space $\mathbb{E}^n$ (where $\mathbb{E}^n$ is the affine space $\mathbb{A}^n$ equipped with its usual inner product on $\mathbb{R}^n$), given any $k \in \mathbb{R}$ with $k \neq 0$ and any point a, an *inversion of pole a and power k* is a map $h\colon (\mathbb{E}^n - \{a\}) \to \mathbb{E}^n$ defined such that for every $x \in \mathbb{E}^n - \{a\}$,

$$h(x) = a + k\frac{\overrightarrow{ax}}{\|\overrightarrow{ax}\|^2}.$$

For example, when $n = 2$, choosing any orthonormal frame with origin a, h is defined by the map

$$(x,y) \mapsto \left(\frac{kx}{x^2+y^2}, \frac{ky}{x^2+y^2}\right).$$

(a) Assuming for simplicity that $n = 2$, viewing $\mathbb{RP}^2$ as the projective completion of $\mathbb{E}^2$, we can extend h to a partial map $h\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ as follows. Pick any projective frame (a_0, a_1, a_2, a_3) where $a_0 = a + e_1$, $a_1 = a + e_2$, $a_2 = a$, $a_3 = a + e_1 + e_2$, and where (e_1, e_2) is an orthonormal basis for $\mathbb{R}^2$, and define h such that in homogeneous coordinates

$$(x,y,z) \mapsto (kxz, kyz, x^2+y^2).$$

Show that h is defined on $\mathbb{RP}^2 - \{a\}$. Show that $h \circ h = \mathrm{id}$, except for points on the line at infinity (that are all mapped onto $a = (0,0,1)$). Deduce that h is a bijection except for a and the points on the line at infinity. Show that the fixed points of h are on the circle of equation

$$x^2 + y^2 = kz^2.$$

(b) We can also extend h to a partial map $h\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ as in the real case, and define h such that in homogeneous (complex) coordinates

$$(x,y,z) \mapsto (kxz, kyz, x^2+y^2).$$

Show that h is defined on $\mathbb{CP}^2 - \{a, I, J\}$, where $I = (1, -\mathrm{i}, 0)$ and $J = (1, \mathrm{i}, 0)$ are the circular points. Show that every point of the line $\langle I, J\rangle$ other than I and J is mapped to A, every point of the line $\langle A, I\rangle$ other than A and I is mapped to I, and every point of the line $\langle A, J\rangle$ other than A and J is mapped to J. Show that $h \circ h = \mathrm{id}$ on the complement of the three lines $\langle I, J\rangle$, $\langle A, I\rangle$, and $\langle A, J\rangle$. Show that the fixed points of h are on the circle of equation

$$x^2 + y^2 = kz^2.$$

Say that a circle of equation

$$ax^2 + ay^2 + bxz + cyz + dz^2 = 0$$

is a *true circle* if $a \neq 0$. We define the *center* of a circle as above (true or not) as the point of homogeneous coordinates $(b, c, -2a)$ and the *radius R* of a true circle is defined as follows: If

$$b^2 + c^2 - 4ad > 0,$$

then $R = \sqrt{b^2+c^2-4ad}/(2a)$; otherwise $R = \mathrm{i}\sqrt{4ad-b^2-c^2}/(2a)$. Note that R can be a complex number. Also, when $a = 0$, we let $R = \infty$.

Verify that in the affine Euclidean plane $\mathbb{E}^2$ (the complement of the line at infinity $z = 0$) the notions of center and radius have the usual meaning (when R is real).

(c) Show that the image of a circle of equation

$$ax^2 + ay^2 + bxz + cyz + dz^2 = 0$$

is the circle of equation

$$dx^2 + dy^2 + kbxz + kcyz + k^2az^2 = 0.$$

When does a true circle map to a true circle?

(d) Recall the definition of the *stereographic projection map* $\sigma\colon (S^2 - \{N\}) \to \mathbb{R}^2$ from Problem 5.3. Prove that the stereographic projection map is the restriction to S^2 of an inversion of pole N and power $2R^2$ in $\mathbb{E}^3$ (where S^2 a sphere of radius R, N is the north pole of S^2, and the plane of projection is a plane through the center of the sphere).

5.38. As in Problem 5.37, we consider inversions in $\mathbb{RP}^2$ and $\mathbb{CP}^2$, and we assume that some projective frame (a_0, a_1, a_2, a_3) is chosen.

(a) Given two distinct real circles C_1 and C_2 of equations

$$\begin{aligned} x^2 + y^2 - R^2z^2 &= 0, \\ x^2 + y^2 - 2bxz + dz^2 &= 0, \end{aligned}$$

prove that C_1 and C_2 intersect in two real points iff the line

$$2bx - (d + R^2)z = 0$$

intersects C_1 in two real points iff

$$(R^2 + d - 2bR)(R^2 + d + 2bR) < 0.$$

The line $2bx - (d + R^2)z = 0$ is called the *radical axis D* of the circles C_1 and C_2. If $b = 0$, then C_1 and C_2 have the same center, and the radical axis is the line at infinity. Otherwise, if $b \neq 0$, by chosing a new frame (b_0, b_1, b_2, b_3) such that

$$b_0 = \left(\frac{R^2+d}{2b} + 1, 0, 0\right), \quad b_1 = \left(\frac{R^2+d}{2b}, 1, 0\right), \quad b_2 = \left(\frac{R^2+d}{2b}, 0, 1\right),$$

and

$$b_3 = \left(\frac{R^2+d}{2b}, 1, 1\right),$$

show that the equations of the circles C_1, C_2 become

$$4b^2(x^2+y^2)+4b(R^2+d)xz+\Delta z^2 = 0,$$
$$4b^2(x^2+y^2)+4b(R^2+d-2b^2)xz+\Delta z^2 = 0,$$

where $\Delta = (R^2+d-2bR)(R^2+d+2bR)$.

Letting $C = \Delta/(4b^2)$, the above equations are of the form

$$x^2+y^2-2uxz+Cz^2 = 0,$$
$$x^2+y^2-2vxz+Cz^2 = 0,$$

where $u \neq v$.

(b) Consider the pencil of circles defined by C_1 and C_2, i.e., the set of all circles having an equation of the form

$$(\lambda+\mu)(x^2+y^2)-2(\lambda u+\mu v)xz+(\lambda+\mu)Cz^2 = 0,$$

where $(\lambda,\mu) \neq (0,0)$.

If $C < 0$, letting $K^2 = -C$ where $K > 0$, prove that the circles in the pencil are exactly the circles passing through the points $A = (0,K,1)$ and $B = (0,-K,1)$, called *base points* of the pencil. In this case, prove that the image of all the circles in the pencil by an inversion h of center A is the union of the line at infinity together with the set of all lines through the image $h(B)$ of B under the inversion (pick a convenient frame).

(c) If $C = 0$, in which case $A = B = (0,0,1)$, prove that the circles in the pencil are exactly the circles tangent to the radical axis D (at the origin). In this case, prove that the image of all the circles in the pencil by an inversion h of center A is the union of the line at infinity together with the set of all lines parallel to the radical axis D.

(d) If $C > 0$, letting $K^2 = C$ where $K > 0$, prove that there exist two circles in the pencil of radius 0 and of centers $P_1 = (K,0,1)$ and $P_2 = (-K,0,1)$, called the *Poncelet points* of the pencil. In this case, prove that the image of all the circles in the pencil by an inversion of center P_1 is the set of all circles of center $h(P_2)$ (pick a convenient frame).

Conclude that given any two distinct nonconcentric real circles C_1 and C_2, there is an inversion such that if C_1 and C_2 intersect in two real points, then C_1 and C_2 are mapped to two lines (plus the line at infinity), and if C_1 and C_2 are disjoint (as real circles), then C_1 and C_2 are mapped to two concentric circles.

(e) Given two C^1-curves Γ,Δ in $\mathbb{E}^2$, if Γ and Δ intersect in p, prove that for any inversion h of pole $c \neq p$, h preserves the absolute value of the angle of the tangents to Γ and Δ at p. Conclude that inversions preserve tangency and orthogonality.
Hint. Express Γ,Δ, and h in polar coordinates.

(f) Using (e), prove the following beautiful theorem of Steiner. Let C_1 and C_2 be two disjoint real circles such that C_2 is inside C_1. Construct any sequence $(\Gamma_n)_{n\geq 0}$ of circles such that Γ_n is any circle interior to C_1, exterior to C_2, tangent to C_1 and C_2, and furthermore that $\Gamma_{n+1} \neq \Gamma_{n-1}$ and Γ_{n+1} is tangent to Γ_n.

Given a starting circle Γ_0, two cases may arise: Either $\Gamma_n = \Gamma_0$ for some $n \geq 1$, or $\Gamma_n \neq \Gamma_0$ for all $n \geq 1$.

Prove that the outcome is independent of the starting circle Γ_0. In other words, either for every Γ_0 we have $\Gamma_n = \Gamma_0$ for some $n \geq 1$, or for every Γ_0 we have $\Gamma_n \neq \Gamma_0$ for all $n \geq 1$.

5.39. (a) Let $h\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ be the projectivity (w.r.t. any projective frame (a_0, a_1, a_2, a_3)) defined such that

$$(x,\, y,\, z) \mapsto (x,\, y,\, ax + by + cz),$$

where $c \neq 0$ and h is not the identity.

Prove that the fixed points of h (i.e., those points M such that $h(M) = M$) are the origin $O = a_2 = (0,0,1)$ and every point on the line Δ of equation

$$ax + by + (c-1)z = 0.$$

Prove that every line through the origin is globally invariant under h. Give a geometric construction of $h(M)$ for every point M distinct from O and not on Δ, given any point A distinct from O and not on Δ and its image $A' = h(A)$.
Hint. Consider the intersection P of the line $\langle A, M\rangle$ with the line Δ.

Such a projectivity is called a *homology of center O and of axis* Δ (Poncelet).

Show that in the situation of Desargues's theorem, the triangles (a,b,c) and (a',b',c') are homologous. What is the axis of homology?

(b) Let $h\colon \mathbb{RP}^3 \to \mathbb{RP}^3$ be the projectivity (w.r.t. any projective frame $(a_0, a_1, a_2, a_3, a_4)$) defined such that

$$(x,\, y,\, z,\; t) \mapsto (x,\, y,\, z,\, ax + by + cz + dt),$$

where $d \neq 0$ and h is not the identity.

Prove that the fixed points of h (i.e., those points M such that $h(M) = M$) are the origin $O = a_3 = (0,0,0,1)$ and every point on the plane Π of equation

$$ax + by + cz + (d-1)t = 0.$$

Prove that every line through the origin is globally invariant under h. Give a geometric construction of $h(M)$ for every point M distinct from O and not on Π, given any point A distinct from O and not on Π and its image $A' = h(A)$.
Hint. Consider the intersection P of the line $\langle A, M\rangle$ with the plane Π.

Such a projectivity is called a *homology of center O and of plane of homology* Π (Poncelet).

(c) Let $h\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ be a projectivity, and assume that h does not preserve (globally) the line at infinity $z = 0$. Prove that there is a rotation R and a point at infinity a_1 such that $h \circ R$ maps all lines through a_1 to lines through a_1.

Chosing a projective frame (a_0, a_1, a_2, a_3) (where a_1 is the point mentioned above), show that $h \circ R$ is defined by a matrix of the form

$$\begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & b'' & c'' \end{pmatrix}$$

where $a \neq 0$ and $b'' \neq 0$. Prove that there exist two translations t_1, t_2 such that $t_2 \circ h \circ R \circ t_1$ is a homology.

If h preserves globally the line at infinity, show that there is a translation t such that $t \circ h$ is defined by a matrix of the form

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{pmatrix}$$

where $ab' - a'b \neq 0$. Prove that there exist two rotations R_1, R_2 such that $R_2 \circ t \circ h \circ R_1$ has a matrix of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $AB = ab' - a'b$. Conclude that $R_2 \circ t \circ h \circ R_1$ is a homology only when $A = B$.

Remark: The above problem is adapted from Darboux.

5.40. Prove that every projectivity $h\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ where $h \neq \mathrm{id}$ and h is not a homology is the composition of two homologies.

5.41. Given any two tetrahedra (a,b,c,d) and (a',b',c',d') in $\mathbb{RP}^3$ where a,b,c,d, a',b',c',d' are pairwise distinct and the lines containing the edges of the two tetrahedra are pairwise distinct, if the lines $\langle a,a'\rangle$, $\langle b,b'\rangle$, $\langle c,c'\rangle$, and $\langle d,d'\rangle$ intersect in a common point O distinct from a,b,c,d, a',b',c',d', prove that the intersection points (of lines) $p = \langle b,c\rangle \cap \langle b',c'\rangle$, $q = \langle a,c\rangle \cap \langle a',c'\rangle$, $r = \langle a,b\rangle \cap \langle a',b'\rangle$, $s = \langle c,d\rangle \cap \langle c',d'\rangle$, $t = \langle b,d\rangle \cap \langle b',d'\rangle$, $u = \langle a,d\rangle \cap \langle a',d'\rangle$, are coplanar.

Prove that the lines of intersection (of planes) $P = \langle b,c,d\rangle \cap \langle b',c',d'\rangle$, $Q = \langle a,c,d\rangle \cap \langle a',c',d'\rangle$, $R = \langle a,b,d\rangle \cap \langle a',b',d'\rangle$, $S = \langle a,b,c\rangle \cap \langle a',b',c'\rangle$, are coplanar.

Hint. Show that there is a homology whose center is O and whose plane of homology is determined by p,q,r,s,t,u.

5.42. Prove that Pappus's theorem implies Desargues's theorem (in the plane).

5.43. If K is a finite field of q elements ($q \geq 2$), prove that the finite projective space $\mathbf{P}(K^{n+1})$ has $q^n + q^{n-1} + \cdots + q + 1$ points and

$$\frac{(q^{n+1}-1)(q^n-1)}{(q-1)^2(q+1)}$$

lines.

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