1 Projective Geometry

2D Projective Plane, $P^2$, is the set of an equivalence class of vectors in $\mathbb{R}^3 \setminus \{0\}$. Formally,

$$ P^2 = \{ [a] \mid a \in \mathbb{R}^3 \setminus \{0\} \}, $$

where the equivalence class of an element $a$ is denoted as $[a]$, $[a] = \{ x \in \mathbb{R}^3 \setminus \{0\} \mid a \sim x \}$, and the equivalence relation, $\sim$, is defined as follows:

$$ \begin{bmatrix} x \\ y \\ \omega \end{bmatrix} \sim \begin{bmatrix} x' \\ y' \\ \omega' \end{bmatrix} \text{ iff } \exists \lambda \neq 0 \begin{bmatrix} x \\ y \\ \omega \end{bmatrix} = \lambda \begin{bmatrix} x' \\ y' \\ \omega' \end{bmatrix} \text{ for } \begin{bmatrix} x \\ y \\ \omega \end{bmatrix}, \begin{bmatrix} x' \\ y' \\ \omega' \end{bmatrix} \in \mathbb{R}^3 \setminus \{0\}. $$

The homogeneous representation of a 2D point or line is an element of $P^2$.

1. Properties of a homogeneous representation of a 2D point and line.

   (a) Let a 2D point in an image be $\mathbf{f}$. What is the homogeneous representation, $\mathbf{x}$, of this point?

   (b) A standard line equation in 2D plane is $ax + by + c = 0$. What is the homogeneous representation, $\mathbf{l}$, of this line?

   (c) What is the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$?

   (d) What is the line passing two points $\mathbf{x}$ and $\mathbf{x}'$?

   (e) Points at infinity $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ pass the line at infinity. What is the form of the line at infinity, $\mathbf{L}_\infty$? Why?

2. Finding a projective transformation $H$.

   A projective transformation $H$ preserves the points $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the origin of the coordinate system. However, it maps the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the points $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ meaning $d \begin{bmatrix} 2 \\ 1 \end{bmatrix} = H \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Compute $H$ with one free variable $d$. 

Instructions. All coordinate systems are right handed.
Figure 1: Cross-ratio.

2 Camera Geometry

1. Camera Projection.
   The projection from the world to the image plane in at the camera coordinates is:

   \[
   K^{-1} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \sim \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ t \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix},
   \]

   where \( K \) is an invertible \( 3 \times 3 \) matrix of intrinsic parameters, \( r_1, r_2 \) and \( r_3 \) are the three columns of a rotation matrix, and \( t \) is a translation vector.

   (a) Find the projective transformation, \( H \), from the plane \( Y = 0 \) to the image plane in the camera coordinates.

   (b) Two photographers took a same planar object on the \( Y = 0 \) plane at the two different views. Assume we've estimated the projective transformation \( H_1 \) from the planar object to the image plane in the first camera coordinates and also estimated the projective transformation \( H_2 \) from the planar object to the image plane in the second camera coordinates. Find the camera pose, \( R_1 \) and \( t_1 \), of the first one. Feel free to use Matlab notations.

   (c) Find the distance between two cameras. Assume the rotation and translation of second camera are \( R_2 \) and \( t_2 \), respectively.

3 Single View Metrology

1. Cross-ratio.
   In Figure 1, what is the remaining distance of the car to reach the destination (the checkered flag)? Assume we know that the distance between two wheels, which is \( d \).

4 Intrinsic Calibration

1. Calibration from two vanishing points.
The projection equation from the world to the image is
\[
\begin{bmatrix}
u \\
v \\
1
\end{bmatrix}
\sim
\begin{bmatrix}
f \\ 0 \\ 0 \\
0 \\ f \\ 0 \\
0 \\ 0 \\ 1
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix}.
\]

Argue that you can find the focal length \(f\) from \(x/y\)-vanishing points if the vanishing points do not lie at infinity.


We will verify the following relationship:
\[
[u_{\text{dist}} \ v_{\text{dist}}] = \begin{bmatrix}
 u_{\text{undist}} + (u_{\text{undist}} - p_x)(k_1 r^2 + k_2 r^4) \\
v_{\text{undist}} + (v_{\text{undist}} - p_y)(k_1 r^2 + k_2 r^4)
\end{bmatrix}.
\]

Note that \([u_{\text{dist}} \ v_{\text{dist}}]\) is the measured point in pixel coordinates and \([u_{\text{undist}} \ v_{\text{undist}}]\) is the projected 2D point without considering lens distortion. From this equation, we can related the measured points and the projected 2D points through a radial lens distortion model, which will be used to linearly estimate \(k_1\) and \(k_2\).

(a) Write down the Camera Projection without lens distortion in terms of \(R\) and \(t\). This projection projects the 3D world point \([X \ Y \ Z \ 1]\) to the 2D undistorted image point \([x_{\text{undist}} \ y_{\text{undist}} \ 1]\) in the camera (not pixel) coordinates.

(b) Find \([u_{\text{dist}} \ v_{\text{dist}} \ 1]\) by applying radial lens distortion on \([x_{\text{undist}} \ y_{\text{undist}} \ 1]\) with two parameters \(k_1\) and \(k_2\).

(c) Find \([u_{\text{dist}} \ v_{\text{dist}} \ 1]\) by applying \(K\) such that
\[
\begin{bmatrix}
u_{\text{dist}} \\
v_{\text{dist}} \\
1
\end{bmatrix}
\sim
\begin{bmatrix}
f_x \\ 0 \\ p_x \\
0 \\ f_y \\ p_y \\
0 \\ 0 \\ 1
\end{bmatrix}
\begin{bmatrix}
x_{\text{dist}} \\
y_{\text{dist}} \\
1
\end{bmatrix}.
\]

(d) Express \([u_{\text{dist}} \ v_{\text{dist}} \ 1]\) in terms of \(u_{\text{undist}}\) and \(v_{\text{undist}}\).

Note that
\[
\begin{bmatrix}
u_{\text{undist}} \\
v_{\text{undist}} \\
1
\end{bmatrix}
\sim
\begin{bmatrix}
f_x \\ 0 \\ p_x \\
0 \\ f_y \\ p_y \\
0 \\ 0 \\ 1
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} = \begin{bmatrix}
X_{\text{undist}} \\
Y_{\text{undist}} \\
Z_{\text{undist}} \\
1
\end{bmatrix} = \begin{bmatrix}
f_x x_{\text{undist}} + p_x \\
f_y y_{\text{undist}} + p_y \\
1
\end{bmatrix},
\]
which is a camera projection without considering the lens distortion.