7.8 Orientations of a Euclidean Space, Angles

In this section we return to vector spaces. In order to deal with the notion of orientation correctly, it is important to assume that every family $(u_1, \ldots, u_n)$ of vectors is ordered (by the natural ordering on $\{1, 2, \ldots, n\}$). Thus, we will assume that all families $(u_1, \ldots, u_n)$ of vectors, in particular bases and orthonormal bases, are ordered.

Let E be a vector space of finite dimension n over $\mathbb{R}$, and let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be any two bases for E. Recall that the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$ is the matrix P whose columns are the coordinates of the vectors v_j over the basis $(u_1, \ldots, u_n)$. It is immediately verified that the set of alternating n-linear forms on E is a vector space, which we denote by $\Lambda(E)$ (see Lang [107]).

We now show that $\Lambda(E)$ has dimension 1. For any alternating n-linear form $\varphi: E \times \cdots \times E \to K$ and any two sequences of vectors $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, if

$$(v_1, \ldots, v_n) = (u_1, \ldots, u_n)P,$$

then

$$\varphi(v_1, \ldots, v_n) = \det(P)\varphi(u_1, \ldots, u_n).$$

In particular, if we consider nonnull alternating n-linear forms $\varphi: E \times \cdots \times E \to K$, we must have $\varphi(u_1, \ldots, u_n) \neq 0$ for every basis $(u_1, \ldots, u_n)$. Since for any two alternating n-linear forms φ and ψ we have

$$\varphi(v_1, \ldots, v_n) = \det(P)\varphi(u_1, \ldots, u_n)$$

and

$$\psi(v_1, \ldots, v_n) = \det(P)\psi(u_1, \ldots, u_n),$$

we get

$$\varphi(u_1, \ldots, u_n)\psi(v_1, \ldots, v_n) - \psi(u_1, \ldots, u_n)\varphi(v_1, \ldots, v_n) = 0.$$

Fixing $(u_1, \ldots, u_n)$ and letting $(v_1, \ldots, v_n)$ vary, this shows that φ and ψ are linearly dependent, and since $\Lambda(E)$ is nontrivial, it has dimension 1.

We now define an equivalence relation on $\Lambda(E) - \{0\}$ (where we let 0 denote the null alternating n-linear form):

φ and ψ are equivalent if $\psi = \lambda\varphi$ for some $\lambda > 0$.

It is immediately verified that the above relation is an equivalence relation. Furthermore, it has exactly two equivalence classes O_1 and O_2.

The first way of defining an *orientation of E* is to pick one of these two equivalence classes, say O $(O \in \{O_1, O_2\})$. Given such a choice of a class O, we say that a basis $(w_1, \ldots, w_n)$ has *positive orientation* iff