The g-Theorem
MA715 Course Notes
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1 Introduction

These notes will be rather informal in many places. For more precision, refer to *Lectures on Polytopes* by Günter Ziegler [Zie95], *Convex Polytopes* by Branko Grünbaum [Grü67], *An Introduction to Convex Polytopes* by Arne Brøndsted [Brø83], and *Convex Polytopes and the Upper Bound Conjecture* by Peter McMullen and Geoffrey Shephard [MS71]. See also the handbook papers [BL93, KK95]

We will be studying polyhedra and polytopes. A *(convex) polyhedron* is the intersection of a finite number of closed halfspaces in $\mathbf{R}^d$. A *(convex) polytope* is a bounded polyhedron; equivalently, it is the *convex hull* (the smallest convex set containing) a given finite set of points.

If P is a polyhedron and H is a hyperplane such that P is contained in one of the closed halfspaces associated with H and $F := H \cap P$ is nonempty, then H is a *supporting hyperplane* of P. In such a case, if $F \neq P$, then F is a *proper face* of P. The *improper faces* of P are the empty set and P itself. Let $\mathcal{F}(P)$ denote the set of all faces of P, both proper and improper, and let $\mathcal{F}(\operatorname{bd} P) := \mathcal{F}(P) \setminus \{P\}$.

The *dimension* of a subset of $\mathbf{R}^d$ is the dimension of its affine span. If P is a polyhedron and $\dim P = d$, then P is called a *d-polyhedron*, and faces of P of dimension 0, 1, $d-2$, and $d-1$ are called *vertices*, *edges*, *subfacets* (or *ridges*), and *facets* of P, respectively. We denote the number of j-dimensional faces (*j-faces*) of P by $f_j(P)$ (or simply f_j when the polyhedron is clear) and call $f(P) := (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ the *f-vector* of P. The empty set is the unique face of dimension -1 and P is the unique face of dimension d, so $f_{-1}(P) = 1$ and $f_d(P) = 1$.

The big problem is to understand/describe $f(\mathcal{P}^d) := \{f(P) : P \text{ is a } d\text{-polytope}\}$!

2 Euler's Relation

2.1 Euler's Relation for 3-Polytopes

By now you have all probably encountered the formula "$V - E + F = 2$" for convex three-dimensional polytopes.

Theorem 2.1 (Euler's Relation) *If P is a 3-polytope, then $f_0 - f_1 + f_2 = 2$.*

For historical notes, see [BLW86], [Cro97]. [Grü67], and [Lak76]. Before Euler stated his formula, Descartes discovered a theorem from which Euler's formula could be deduced, but it does not appear that Descartes explicitly did so.

Proof by "immersion". Position the polytope P in a container so that no two vertices are at the same vertical level (have the same z-coordinate). Fill the container with water. Count the contribution of a face to the expression $f_0 - f_1 + f_2$ at the moment when it is submerged. At the very beginning, only the bottom vertex becomes submerged, so at this point $f_0 - f_1 + f_2 = 1 - 0 + 0 = 1$. When a later vertex v (but not the top one) is submerged, that vertex now contributes, as do the downward edges incident to v (let's say there are k of them) and the $k - 1$ facets between these k-edges. So the contribution of these newly submerged faces to $f_0 - f_1 + f_2$ is $1 - k + (k - 1) = 0$. Thus $f_0 - f_1 + f_2$ remains equal to 1. But when the top vertex v is submerged, all of its incident edges (let's say there are k of them) are submerged), as well as k incident facets. The contribution of these newly submerged faces to $f_0 - f_1 + f_2$ is $1 - k + k = 1$, so at the end $f_0 - f_1 + f_2 = 2$. □

Exercise 2.2 What happens when you apply this proof technique to a "polyhedral torus"? □

Proof by "projection and destruction". Choose a facet F of P. Find a point q outside of the polytope P but "sufficiently close" to a point in the relative interior of F. Let H be a plane parallel to F but not containing q. Project the vertices and the edges of P onto H using central projection towards q. Now you have a connected, planar graph G in H, called a *Schlegel diagram* of P. There is a bijection between the regions of H determined by G, and the facets of P. Let f_0, f_1, and f_2 be the number of vertices, edges, and regions, respectively, of G. Find a cycle in G, if there is one (a sequence of vertices and edges $v_0e_1v_1e_2v_2 \cdots v_ke_k$ where $k > 2$, the v_i are distinct, and e_i joins v_{i-1} to v_i, $i = 1, \ldots, k$). Delete one edge of the cycle. Then f_1 and f_2 each drop by one (why?). So $f_0 - f_1 + f_2$ does not change. Repeat this step until no cycles remain. Now find a vertex incident to precisely one of the remaining edges (why does the nonexistence of cycles imply that such a vertex exists?). Delete this vertex and this edge. Then f_0 and f_1 each drop by one. So $f_0 - f_1 + f_2$ does not change. Repeat this step until the graph is reduced to a single vertex and a single region with no edges (this is where the connectivity of G comes into play). At this stage $f_0 - f_1 + f_2 = 1 - 0 + 1 = 2$, so it must have been equal to two at the start as well. $\square$

This proof applies to arbitrary connected planar graphs.

Exercise 2.3 What is "sufficiently close" in the above proof? $\square$

Proof by "shelling". Build up the boundary of the polytope facet by facet, keeping track of $f_0 - f_1 + f_2$ as you go. Be sure each new facet F (except the last) meets the union of the previous ones in a single path of vertices and edges along the boundary of F. Suppose the first facet has k edges. Then at this point $f_0 - f_1 + f_2 = k - k + 1 = 1$. Suppose a later facet F (but not the last) has k edges but meets the previous facets along a path with ℓ edges and $\ell + 1$ vertices, $\ell < k$. Then F as a whole increases $f_0 - f_1 + f_2$ by $k - k + 1 = 1$, but we must subtract off the contribution by the vertices and edges in the intersection, which is $(\ell + 1) - \ell = 1$. So there is no net change to $f_0 - f_1 + f_2$. The very last facet increases only f_2 (by one), giving the final result $f_0 - f_1 + f_2 = 2$. $\square$

At this point, however, it is not obvious that every 3-polytope can be built up in such a way, so this proof requires more work to make it secure.

Proof by algebra. Let $\mathcal{F}_j$ denote the set of all j-faces of P, $j = -1, 0, 1, 2$. For $j = -1, 0, 1, 2$ define vector spaces $X_j = \mathbf{Z}_2^{\mathcal{F}_j}$ over $\mathbf{Z}_2$ with coordinates indexed by the j-faces. If you like, you may think of a bijection between the vectors of X_j and the subsets of $\mathcal{F}_j$. In particular, $\dim X_j = f_j$. For $j = 0, 1, 2$ we are going to define a linear *boundary* map $\partial_j : X_j \to X_{j-1}$. Assume $x = (x_F)_{F \in \mathcal{F}_j}$. Let $\partial_j(x) = (y_G)_{G \in \mathcal{F}_{j-1}}$ be defined by

$$y_G = \sum_{F : G \subset F} x_F.$$

Define also $\partial_{-1} : X_{-1} \to 0$ by $\partial_{-1}(x) = 0$, and $\partial_3 : 0 \to X_2$ by $\partial_3(0) = 0$. You should be able to verify that $\partial_{j-1}\partial_j(x)$ equals zero for all $x \in X_j$, $j = 0, 1, 2$ (why?). Set $B_j = \partial_{j+1}(X_{j+1})$ and $C_j = \ker \partial_j$, $j = -1, 0, 1, 2$. By the previous observation, $B_j \subseteq C_j$. The subspaces B_j are called *j-boundaries* and the subspaces C_j are called *j-cycles*. Note that $\dim X_j = \dim B_{j-1} + \dim C_j$, $j = 0, 1, 2$. Finally define the quotient spaces $H_j = C_j / B_j$, $j = -1, \ldots, 2$. (These are the *(reduced) homology spaces* of the boundary complex of P over $\mathbf{Z}_2$.) Then $\dim H_j = \dim C_j - \dim B_j$. It turns out that B_j actually equals C_j, $j = -1, 0, 1$

(this is not obvious), so for these values of j we have $\dim B_j = \dim C_j$ and $\dim H_j = 0$. Observe that $\dim B_2 = 0$ and $\dim C_2 = 1$ (why?). So $\dim H_2 = 1$, and we have

$$\begin{aligned}
1 &= \dim H_2 - \dim H_1 + \dim H_0 - \dim H_{-1} \\
&= (\dim C_2 - \dim B_2) - (\dim C_1 - \dim B_1) + (\dim C_0 - \dim B_0) - (\dim C_{-1} - \dim B_{-1}) \\
&= -\dim B_2 + (\dim C_2 + \dim B_1) - (\dim C_1 + \dim B_0) + (\dim C_0 + \dim B_{-1}) - \dim C_{-1} \\
&= 0 + \dim X_2 - \dim X_1 + \dim X_0 - 1 \\
&= f_2 - f_1 + f_0 - 1.
\end{aligned}$$

This implies $2 = f_2 - f_1 + f_0$. □

Exercise 2.4 If P is a 3-polytope, prove that $\partial_{j-1}\partial_j(x) = 0$ equals zero for all $x \in X_j$, $j = 0, 1, 2$. □

Exercise 2.5 Begin thinking about which of the above proofs might generalize to higher dimensions, and how. □

2.2 Some Consequences of Euler's Relation for 3-Polytopes

Exercise 2.6 For a 3-polytope P, let p_i denote the number of faces that have i vertices (and hence i edges), $i = 3, 4, 5, \ldots$. (The vector $(p_3, p_4, p_5, \ldots)$ is called the *p-vector* of P.) Let q_i denote the number of vertices at which i faces (and hence i edges) meet, $i = 3, 4, 5, \ldots$.

1. Prove
   (a) $3p_3 + 4p_4 + 5p_5 + 6p_6 + \cdots = 2f_1$.
   (b) $2f_1 \geq 3f_2$.
   (c) $3q_3 + 4q_4 + 5q_5 + 6q_6 + \cdots = 2f_1$.
   (d) $2f_1 \geq 3f_0$.
   (e) $f_2 \leq 2f_0 - 4$.
   (f) $f_2 \geq \frac{1}{2}f_0 + 2$.
2. Label the horizontal axis in a coordinate system f_0 and the vertical axis f_2. Graph the region for which the above two inequalities (1e) and (1f) hold.
3. Consider all integral points (f_0, f_2) lying in the above region. Can you find a formula for the number of different possible values of integral values f_2 for a given integral value of f_0?

4. Prove that no 3-polytope has exactly 7 edges.

5. Think of ways to construct 3-polytopes that achieve each possible integral point (f_0, f_2) in the region.

6. Prove that $f_0 - f_1 + f_2 = 2$ is the unique linear equation (up to nonzero multiple) satisfied by the set of f-vectors of all 3-polytopes.

7. **Characterization of $f(\mathcal{P}^3)$.** Describe necessary and sufficient conditions for (f_0, f_1, f_2) to be the f-vector of a 3-polytope.

□

Exercise 2.7

1. Prove the following inequalities for 3-polytopes.

   (a) $6 \leq 3f_0 - f_1$.

   (b) $6 \leq 3f_2 - f_1$.

   (c) $12 \leq 3p_3 + 2p_4 + 1p_5 + 0p_6 - 1p_7 - 2p_8 - \cdots$.

2. Prove that every 3-polytope must have at least one face that is a triangle, quadrilateral, or pentagon.

3. Prove that every 3-polytope must have at least one vertex at which exactly 3, 4, or 5 edges meet.

4. A truncated icosahedron (soccer ball) is an example of a 3-polytope such that (1) each face is a pentagon or a hexagon, and (2) exactly three faces meet at each vertex. Prove that any 3-polytope with these two properties must have exactly 12 pentagons.

□

Exercise 2.8 Suppose P is a 3-polytope with the property that each facet has exactly n edges and exactly m edges meet at each vertex. (The *Platonic* (or *regular*) *solids* satisfy these criteria.) List all the possible pairs (m, n). □

Exercise 2.9 Suppose P is a 3-polytope with the property that exactly a_k k-gons meet at each vertex, $k = 3, \ldots, \ell$. (The *semiregular solids*, including the *Archimedean solids*, satisfy this criterion.) Determine f_0, f_1, and f_2 in terms of $a_3, \ldots, a_\ell$. □

Exercise 2.10 Recall from plane geometry that for any polygon, the sum of the exterior angles (the amount by which the interior angle falls short of π) always equals 2π. There is a similar formula for 3-polytopes. For each vertex calculate by how much the sum of the interior angles of the polygons meeting there falls short of 2π. Then sum these shortfalls over all the vertices. Prove that this sum equals 4π. $\square$

2.3 Euler's Relation in Higher Dimensions

Grünbaum [Grü67] credits Schläfli [Sch01] for the discovery of Euler's Relation for d-polytopes in 1852 (though published in 1902). He explains that there were many other discoveries of the relation in the 1880's, but these relied upon the unproven assumption that the boundary complexes of polytopes were suitably "shellable." The first real proof seems to be by Poincaré [Poi93, Poi99] in 1899 during the time when the Euler characteristic of manifolds was under development. Perhaps the first completely elementary proof without algebraic overtones is that of Grünbaum [Grü67]. The proof that we give below is a bit different, but still a sibling of Grünbaum's proof.

Theorem 2.11 (Euler-Poincaré Relation) *If P is a d-polytope, then*

$$\chi(P) := \sum_{j=0}^{d-1} (-1)^j f_j(P) = 1 - (-1)^d.$$

The subset $\{(f_0, \ldots, f_{d-1}) \in \mathbf{R}^d : \sum_{j=1}^{d-1} (-1)^j f_j = 1 - (-1)^d\}$ is sometimes called the *Euler hyperplane*.

Two alternative expressions of this result are

$$\hat{\chi}(P) := \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(P) = 1,$$

and

$$\sum_{j=-1}^{d} (-1)^j f_j(P) = 0.$$

Proof. Assume that P is a subset of $\mathbf{R}^d$. Choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for each vertex v of P (why can this be done?). Order the vertices of P, $v_1, \ldots, v_n$, by increasing value of $c^T v_i$. For $k = 1, \ldots, n$, define $S_k(P) := \{F \subset P : F$ is a face of P such that $c^T x \leq c^T v_k$ for all $x \in F\}$. (Clearly $S_n(P) = \mathcal{F}(\operatorname{bd} P)$, the set of all faces of P.) We will prove that

$$\hat{\chi}(S_k(P)) = \begin{cases} 0, & k = 1, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

Our proof is by double induction on d and n. It is easy to check its validity for $d = 0$ and $d = 1$, so fix $d \geq 2$. When $k = 1$, $S_1(P)$ consists of the empty set and v_1, so $\hat{\chi}(S_1(P)) = 0$. Assume $k \geq 2$. Then

$$\begin{aligned}
\hat{\chi}(S_k(P)) &= \hat{\chi}(S_{k-1}(P)) + \hat{\chi}(S_k(P) \setminus S_{k-1}(P)) \\
&= \hat{\chi}(S_k(P) \setminus S_{k-1}(P)).
\end{aligned}$$

Let Q be a *vertex figure* of P at v_k. This is constructed by choosing a hyperplane H for which v_k and the set $\{v_1, \ldots, v_n\} \setminus v_k$ are in opposite open halfspaces associated with H. Then define $Q := P \cap H$. Let $m := f_0(Q)$. It is a fact that Q is a $(d-1)$-polytope, and there is a bijection between the j-faces F of P containing v_k and the $(j-1)$-faces $F \cap H$ of Q. Moreover, the faces in the set $S_k(P) \setminus S_{k-1}(P)$ correspond to the faces in $S_\ell(Q)$, defined using the same vector c, for some $\ell \leq m$, with $\ell < m$ if and only if $k < n$. (You may need to perturb H slightly to ensure that $c^T x$ is different for each vertex of Q.) Therefore

$$\begin{aligned}
\hat{\chi}(S_k(P) \setminus S_{k-1}(P)) &= \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(S_k(P) \setminus S_{k-1}(P)) \\
&= \sum_{j=0}^{d-1} (-1)^{d-j-1} f_j(S_k(P) \setminus S_{k-1}(P)) \\
&= \sum_{j=0}^{d-1} (-1)^{d-j-1} f_{j-1}(S_\ell(Q)) \\
&= \sum_{j=-1}^{d-2} (-1)^{d-j-2} f_j(S_\ell(Q)) \\
&= \hat{\chi}(S_\ell(Q)) \\
&= \begin{cases} 0, & \ell < m, \\ 1, & \ell = m. \end{cases}
\end{aligned}$$

□

If we are looking for linear equations satisfied by members of $f(\mathcal{P}^d)$, we are done:

Theorem 2.12 *Up to scalar multiple, the relation* $\chi(P) = 1 - (-1)^d$ *is the unique linear equation satisfied by all* $(f_0, \ldots, f_{d-1}) \in f(\mathcal{P}^d)$, $d \geq 1$.

PROOF. We prove this by induction on d. For $d = 1$, the relation states $f_0 = 2$, and the result is clear. Assume $d \geq 2$. Suppose $\sum_{j=0}^{d-1} a_j f_j = b$ is satisfied by all $f \in f(\mathcal{P}^d)$, where not all a_j are zero. Let Q be any $(d-1)$-polytope and suppose $f(Q) = (\hat{f}_0, \dots, \hat{f}_{d-2})$. Let P_1 be a *pyramid* over Q and P_2 be a *bipyramid* over Q. Such polytopes are created by first realizing Q as a subset of $\mathbf{R}^d$. The pyramid P_1 is constructed by taking the convex hull of Q and any particular point not in the affine span of Q. The bipyramid P_2 is constructed by taking the convex hull of Q and any particular line segment L such that the intersection of Q and L is a point in the relative interiors of both Q and L. It is a fact that

$$\begin{aligned} f(P_1) &= (\hat{f}_0 + 1, \hat{f}_1 + \hat{f}_0, \hat{f}_2 + \hat{f}_1, \dots, \hat{f}_{d-2} + \hat{f}_{d-3}, 1 + \hat{f}_{d-2}), \\ f(P_2) &= (\hat{f}_0 + 2, \hat{f}_1 + 2\hat{f}_0, \hat{f}_2 + 2\hat{f}_1, \dots, \hat{f}_{d-2} + 2\hat{f}_{d-3}, 2\hat{f}_{d-2}). \end{aligned}$$

Both P_1 and P_2 are d-polytopes, so

$$\sum_{j=0}^{d-1} a_j f_j(P_1) = b,$$

$$\sum_{j=0}^{d-1} a_j f_j(P_2) = b.$$

Subtracting the first equation from the second yields

$$a_0 + a_1\hat{f}_0 + a_2\hat{f}_1 + a_3\hat{f}_2 + \cdots + a_{d-2}\hat{f}_{d-3} + a_{d-1}(\hat{f}_{d-2} - 1) = 0$$

and so

$$a_1\hat{f}_0 + a_2\hat{f}_1 + a_3\hat{f}_2 + \cdots + a_{d-2}\hat{f}_{d-3} + a_{d-1}\hat{f}_{d-2} = a_{d-1} - a_0$$

for all $\hat{f} \in f(\mathcal{P}^{d-1})$. This relation cannot be the trivial relation; otherwise $a_1 = \cdots = a_{d-1} = 0$ and $a_{d-1} - a_0 = 0$, which forces $a_j = 0$ for all j. So by induction this relation must be a nonzero scalar multiple of

$$\hat{f}_0 - \hat{f}_1 + \hat{f}_2 - \cdots + (-1)^{d-2}\hat{f}_{d-2} = 1 - (-1)^{d-1}.$$

Thus $a_1 \neq 0$, $a_j = (-1)^{j-1}a_1$, $j = 1, \dots, d-1$, and $a_{d-1} - a_0 = (1 - (-1)^{d-1})a_1$, so

$$\begin{aligned} a_0 &= a_{d-1} - (1 - (-1)^{d-1})a_1 \\ &= (-1)^{d-2}a_1 - a_1 + (-1)^{d-1}a_1 \\ &= -a_1. \end{aligned}$$

From this we see that $a_j = (-1)^j a_0$, $j = 0, \dots, d-1$, which in turn forces $b = (1 - (-1)^d)a_0$. Therefore $\sum_{j=1}^{d-1} a_j f_j = b$ is a nonzero scalar multiple of Euler's Relation. $\square$

2.4 Gram's Theorem

We now turn to an interesting geometric relative of Euler's Relation. Gram's Theorem is described in terms of solid angle measurement in [Grü67]; in which the history of the theorem and its relatives is discussed (Gram's contribution is for $d = 3$). The form we give here, and its consequence for volume computation, is summarized from Lawrence [Law91a]. See also [Law91b, Law97].

Suppose P is a d-polytope in $\mathbf{R}^d$. Each facet F_i has a unique supporting hyperplane H_i. Let H_i^+ be the closed halfspace associated with H_i containing P.

For every face F, whether proper or not, define

$$K_F := \bigcap_{i:F\subseteq H_i} H_i^+.$$

Note in particular that $K_\emptyset = P$ and $K_P = \mathbf{R}^d$. Define the function $a_F : \mathbf{R}^d \to \mathbf{R}$ by

$$a_F(x) = \begin{cases} 1, & x \in K_F, \\ 0, & x \notin K_F. \end{cases}$$

Theorem 2.13 *If P is a d-polytope, then*

$$\sum_{F:-1\leq \dim F\leq d} (-1)^{\dim F} a_F(x) = 0 \textit{ for all } x \in \mathbf{R}^d.$$

Equivalently,

$$\sum_{F:0\leq \dim F\leq d} (-1)^{\dim F} a_F(x) = \begin{cases} 1, & x \in P, \\ 0, & x \notin P. \end{cases}$$

The proof that the above sum equals one when $x \in P$ follows easily from Euler's Relation. The case $x \notin P$ is more easily understood after we have discussed shellability. □

Every vertex of a d-polytope P must be contained in at least d facets, and also in at least d edges. A d-polytope P is *simple* if every vertex of P is contained in exactly d facets (equivalently, exactly d edges). In linear programming terminology, such polytopes are said to be *nondegenerate*.

Let P be a simple d-polytope in $\mathbf{R}^d$, and again choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for each vertex v of P. For each vertex v let $v_1, \ldots, v_d$ be its neighbors (connected to v by edges) and define $w_i = v_i - v$, $i = 1, \ldots, d$. The vector w_i can be thought of as a vector pointing from v to v_i. Define also the numbers $d_i = c^T v - c^T v_i$, $i = 1, \ldots, d$. So d_i is negative when w_i points "upward" with respect to c, and positive when w_i points "downward" with respect to c. Let $s_v := \text{card}\,\{i : d_i < 0\}$. Construct the cone

$$\hat{K}_v := \{v + \sum_{i=1}^{d} \lambda_i w_i : \lambda_i \geq 0 \text{ if } d_i > 0, \text{ and } \lambda_i < 0 \text{ if } d_i < 0\}.$$

and the associated function

$$\hat{a}_v(x) := \begin{cases} 1, & x \in \hat{K}_v, \\ 0, & x \notin \hat{K}_v. \end{cases}$$

Theorem 2.14 (Lawrence 1991) *Let P be a d-polytope in $\mathbf{R}^d$. Then*

$$\sum_v (-1)^{s_v} \hat{a}_v(x) = \begin{cases} 1, & x \in P, \\ 0, & x \notin P. \end{cases}$$

PROOF. (Sketch.) For vertex $v = v_k$ let S_k be defined as in the previous theorem, but using the vector $-c$ instead of c. Then

$$(-1)^{s_v} \hat{a}_v(x) = \sum_{F \in S_k \setminus S_{k-1} : 0 \leq \dim F \leq d} (-1)^{\dim F} a_F(x).$$

□

If we choose a number y such that $c^T y \leq c^T x$ for all $x \in P$, we can truncate the cone $\hat{K}_v$ by defining $\overline{K}_v := \hat{K}_v \cap \{x : c^T x \geq y\}$. Each of these cones has finite volume, and so Lawrence's theorem gives a way of computing the volume of P in terms of additions and subtractions of the volumes of the $\overline{K}_v$. Lest this sound too complicated, it turns out to be quite easy if we have the simplex tableau associated with each vertex of P, as Lawrence explains.

Without loss of generality, assume that P is contained in the nonnegative orthant of $\mathbf{R}^d$. Choose an objective function c^T that is nonconstant on every edge of P. Lawrence uses the example

$$\begin{array}{lrcl} \text{maximize} & x_1 + x_2 & & \\ \text{subject to} & -x_1 + x_2 & \leq & 2, \\ & x_2 & \leq & 4, \\ & 3x_1 + 2x_2 & \leq & 15, \\ & x_1, x_2 & \geq & 0. \end{array}$$

Introduce slack variables and create the tableau for each of the vertices of P. For each tableau compute

$$N_v := \frac{1}{d!} \frac{1}{\delta_v} \frac{\overline{z}^d}{\overline{c}_{i_1} \cdots \overline{c}_{i_d}}$$

where δ_v is the determinant of the basis matrix associated with v, $\overline{z}$ is the objective function value associated with v, and $c_{i_1}, \ldots, c_{i_d}$ are the nonzero reduced costs associated with v. Then the volume of P is the sum of the N_v.

Here is Lawrence's example:

$$(I)\quad \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 4 \\ 3 & 2 & 0 & 0 & 1 & 15 \\ \hline -1 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \quad N_v = \frac{1}{2!}\frac{1}{1}\frac{0^2}{(-1)(-1)}.$$

$$(II)\quad \left[\begin{array}{ccccc|c} 0 & 5/3 & 1 & 0 & 1/3 & 7 \\ 0 & 1 & 0 & 1 & 0 & 4 \\ 1 & 2/3 & 0 & 0 & 1/3 & 5 \\ \hline 0 & -1/3 & 0 & 0 & 1/3 & 5 \end{array}\right] \quad N_v = \frac{1}{2!}\frac{1}{3}\frac{5^2}{(-1/3)(1/3)}.$$

$$(III)\quad \left[\begin{array}{ccccc|c} 0 & 0 & 1 & -5/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2/3 & 1/3 & 7/3 \\ \hline 0 & 0 & 0 & 1/3 & 1/3 & 19/3 \end{array}\right] \quad N_v = \frac{1}{2!}\frac{1}{3}\frac{(19/3)^2}{(1/3)(1/3)}.$$

$$(IV)\quad \left[\begin{array}{ccccc|c} 0 & 0 & 3 & -5 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 4 \\ 1 & 0 & -1 & 1 & 0 & 2 \\ \hline 0 & 0 & -1 & 2 & 0 & 6 \end{array}\right] \quad N_v = \frac{1}{2!}\frac{1}{1}\frac{6^2}{(-1)(2)}.$$

$$(V)\quad \left[\begin{array}{ccccc|c} 5 & 0 & -2 & 0 & 1 & 11 \\ -1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 1 & 0 & 2 \\ \hline -2 & 0 & 1 & 0 & 0 & 2 \end{array}\right] \quad N_v = \frac{1}{2!}\frac{1}{1}\frac{2^2}{(-2)(1)}.$$

$$\text{Volume (Area)} = \sum N_v = \frac{38}{3}.$$

Exercise 2.15 Let $P(\hat{b})$ be a simple d-polytope defined by linear inequalities $P(\hat{b}) := \{x \in \mathbf{R}^d : a_i^T x \leq \hat{b}_i,\ i = 1, \ldots, n\}$. Choose numbers b_i, $i = 1, \ldots, n$, sufficiently close to $\hat{b}_i$, $i = 1, \ldots, n$, respectively, so that $P(b)$ and $P(\hat{b})$ have the same combinatorial structure. In particular, $P(b)$ is also simple and has the same set of feasible basis matrices as $P(\hat{b})$. Prove that $\mathrm{vol}\,(P(b))$ is a homogeneous polynomial of degree d in the variables $b_1, \ldots, b_n$. $\square$

3 The Dehn-Sommerville Equations

3.1 3-Polytopes

If P is a 3-polytope, then of course $f_0 - f_1 + f_2 = 2$. But if every facet of P is a triangle, then we can say more: $3f_2 = 2f_1$. These two equations are linearly independent, and every equation satisfied by f-vectors of all such 3-polytopes is a consequence of these two.

Exercise 3.1 Prove that the set of f-vectors of 3-polytopes, all of whose facets are triangles, is $\{(f_0, 3f_0 - 6, 2f_0 - 4) : f_0 \in \mathbf{Z},\ f_0 \geq 4\}$. $\square$

What is the situation in higher dimensions?

3.2 Duality and Polarity

It is a fact that for every d-polytope P there is another d-polytope P^* that is *dual* to P, in the sense that there is a inclusion-reversing bijection between the faces of P and the faces of P^*. Under this correspondence, j-faces of P correspond to $(d-j-1)$-faces of P^*, $j = -1, \ldots, d$, and the face lattices of P and P^* are *anti-isomorphic*. In particular, vertices of P correspond to facets of P^*, and vice versa. One way to construct such a polytope P^* is to realize P as a subset of $\mathbf{R}^d$ containing the origin in its interior. Then take P^* to be $\{x \in \mathbf{R}^d : x^T y \leq 1 \text{ for all } y \in P\}$. Equivalently, $P^* = \{x \in \mathbf{R}^d : x^T y \leq 1 \text{ for every vertex } y \text{ of } P\}$. If P^* is defined this way, it is called the *polar dual* or simply the *polar* of P (with respect to the origin).

Exercise 3.2

1. Determine the inequalities for and sketch P^* if P is an octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

2. Repeat if the octahedron P above is translated by the vector $(0, 0, -1/2)$.

3. What happens to P^* in the limit if P is translated by the vector $(0, 0, -t)$ as t approaches 1?

$\square$

A *j-simplex* is a j-dimensional polytope that has exactly $j+1$ vertices. (Note that no j-polytope can have fewer than $j+1$ vertices.) A d-polytope P is called *simplicial* if every proper j-face of P is a j-simplex. Equivalently, it is enough to know that every facet of P is a $(d-1)$-simplex.

Simplicial polytopes are dual to simple polytopes (why?), and $(a_0, \ldots, a_{d-1})$ is the f-vector of some simplicial d-polytope if and only if $(a_{d-1}, \ldots, a_0)$ is the f-vector of some simple d-polytope (why?). Our goal in this section is to learn more about $f(\mathcal{P}_s^d)$, the set of f-vectors of the collection $\mathcal{P}_s^d$ of all simplicial polytopes, but it turns out to be easier to view the situation from the simple standpoint first.

3.3 Simple Polytopes

Let v be a vertex of a simple d-polytope Q. Let E be the collection of the d edges of Q containing v. It is a fact that there is a bijection between subsets S of E of cardinality j and j-faces of Q containing v; namely, that unique face of Q containing v and S, but not $E \setminus S$. (Can you see why this won't be true in general if Q is not simple?)

Now assume that Q is a simple d-polytope in $\mathbf{R}^d$, and choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for every vertex v of Q. As in Section 2.3, order the vertices $v_1, \ldots, v_n$ of Q according to increasing value of $c^T x$, and define the sets $S_k := S_k(Q)$. It is a fact that for every nonempty face F of Q there is a unique point of F that maximizes $c^T x$ over all $x \in F$, and that this point is one of the vertices of Q—the unique vertex v_k such that $F \in S_k \setminus S_{k-1}$. Orient each edge uv of Q in the direction of increasing value of $c^T x$; i.e., so that it is pointing from vertex u to vertex v if $c^T u < c^T v$.

Choose a vertex v_k, and assume that there are exactly i edges pointing into v_k (so v_k has *indegree* i and *outdegree* $d-i$). By the above observations, the number of j-faces of $S_k \setminus S_{k-1}$ equals $\binom{i}{j}$. Let h_i^c be the number of vertices of Q with indegree i. Then since each j-face of Q appears exactly once in some $S_k \setminus S_{k-1}$ (necessarily for some vertex v_k of indegree at least i), we see that

$$f_j = \sum_{i=j}^{d} \binom{i}{j} h_i^c, \ j = 0, \ldots, d. \qquad (1)$$

Exercise 3.3 Define the polynomials

$$\hat{f}(Q,t) = \sum_{j=0}^{d} f_j t^j$$

and

$$\hat{h}(Q,t) = \sum_{i=0}^{d} h_i^c t^i.$$

1. Prove $\hat{f}(Q,t) = \hat{h}(Q,t+1)$.
2. Prove $\hat{h}(Q,t) = \hat{f}(Q,t-1)$.

3. Conclude

$$h_i^c = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \ i = 0, \ldots, d. \tag{2}$$

□

The above exercise proves the surprising fact that the numbers h_i^c are independent of the choice of c. In particular, $h_i^{-c} = h_i^c$ for all $i = 0, \ldots, d$. But the vertices of indegree i with respect to $-c$ are precisely the vertices of outdegree $d-i$ with respect to $-c$, hence the vertices of indegree $d-i$ with respect to c. Therefore, $h_i^{-c} = h_{d-i}^c$ for all i. Dispensing with the now superfluous superscript c, we have

$$h_i = h_{d-i}, \ i = 0, \ldots, d, \tag{3}$$

for every simple d-polytope Q. These are the *Dehn-Sommerville Equations* for simple polytopes. We may, if we wish, drop the superscript in equation (2), and use this formula to *define* h_i, $i = 0, \ldots, d$, for simple d-polytopes. The vector $h := (h_0, \ldots, h_d)$ is the *h-vector* of the simple polytope Q.

Exercise 3.4

1. Calculate the h-vector of a 3-cube.

2. Calculate the f-vector and the h-vector for a d-cube with vertices $(\pm 1, \ldots, \pm 1)$. (The *d-cube* is the Cartesian product of the line segment $[-1, 1]$ with itself d-times.) Suggestion: Use induction on d and the fact that every facet of a d-cube is a $(d-1)$-cube.

□

3.4 Simplicial Polytopes

We now return to the simplicial viewpoint. For a simplicial d-polytope P, let Q be a simple d-polytope dual to P. For $i = 0, \ldots, d$,

$$\begin{aligned}
h_i &= h_i(Q) \\
&= h_{d-i}(Q) \\
&= \sum_{k=d-i}^{d} (-1)^{d-i+k} \binom{k}{d-i} f_k(Q) \\
&= \sum_{k=d-i}^{d} (-1)^{d-i+k} \binom{k}{d-i} f_{d-k-1}(P).
\end{aligned}$$

Let $j = d - k$. Then

$$h_i \;=\; \sum_{j=0}^{i} (-1)^{i+j} \binom{d-j}{d-i} f_{j-1}(P),\ i = 0, \ldots, d. \tag{4}$$

We take equation (4) as the *definition* of $h_i(P) := h_i$, $i = 0, \ldots, d$, and let $h(P) := (h_0(P), \ldots, h_d(P))$ be the *h-vector* of the simplicial polytope P. The following two theorems follow immediately.

Theorem 3.5 (Dehn-Sommerville Equations) *If P is a simplicial d-polytope, then $h_i(P) = h_{d-i}(P)$, $i = 0, \ldots, \lfloor (d-1)/2 \rfloor$.*

Theorem 3.6 *If P is a simplicial d-polytope, then $h_i \geq 0$, $i = 0, \ldots, d$.*

In Theorem 3.5, $\lfloor x \rfloor$ is the *greatest integer function*, defined to be $\lfloor x \rfloor := \max\{y : y \leq x$ and y is an integer$\}$.

For a simplicial polytope P, define the polynomials

$$f(P, t) = \sum_{j=0}^{d} f_{j-1} t^j$$

and

$$h(P, t) = \sum_{i=0}^{d} h_i t^i.$$

Exercise 3.7

1. Prove $h(P, t) = (1 - t)^d f(\frac{t}{1-t})$.

2. Prove $f(P, t) = (1 + t)^d h(\frac{t}{1+t})$.

3. Prove
$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i,\ j = 0, \ldots, d. \tag{5}$$

$\square$

Exercise 3.8

1. Find the formulas for h_0, h_1, and h_d in terms of the f_j.

2. Find the formulas for f_{-1}, f_0, and f_{d-1} in terms of the h_i.

3. Prove that $h_0 = h_d$ is equivalent to Euler's Relation for simplicial d-polytopes.

$\square$

Exercise 3.9 Characterize $h(\mathcal{P}_s^3)$; i.e., characterize which vectors (h_0, h_1, h_2, h_3) are h-vectors of simplicial 3-polytopes. $\square$

Exercise 3.10 Show that the number of monomials of degree s in at most r variables is $\binom{r+s-1}{s}$. $\square$

Exercise 3.11 Show that the number of monomials of degree s in exactly r variables (i.e., each variable appears with positive power) is $\binom{s-1}{r-1}$. $\square$

Exercise 3.12 Use Exercise 3.10 to show that the coefficient of t^s in the expansion of $\frac{1}{(1-t)^r} = (1+t+t^2+\cdots)^r$ is $\binom{r+s-1}{s}$. $\square$

Exercise 3.13

Prove that $f(P, \frac{t}{1-t})$ formally expands to the series $\sum_{\ell=0}^{\infty} H_\ell(P)t^\ell$ where

$$H_\ell(P) = \begin{cases} 1, & \ell = 0, \\ \sum_{j=0}^{\ell-1} f_j(P)\binom{\ell-1}{j}, & \ell > 0, \end{cases}$$

(taking $f_j(P) = 0$ if $j \geq d$). $\square$

Exercise 3.14 Prove Stanley's observation that the f-vector can be derived from the h-vector by constructing a triangle in a manner similar to Pascal's triangle, but replacing the right-hand side of the triangle by the h-vector. The f-vector emerges at the bottom. Consider the example of the octahedron.

$$\begin{array}{ccccccccc}
 & & & & \mathbf{1} & & & & \\
 & & & 1 & & \mathbf{3} & & & \\
 & & 1 & & 4 & & \mathbf{3} & & \\
 & 1 & & 5 & & 7 & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{6} & & \mathbf{12} & & \mathbf{8} & &
\end{array}$$

By subtracting instead of adding, one can convert the f-vector to the h-vector in a similar way. $\square$

Exercise 3.15 What are the f-vector and the h-vector of a d-simplex? $\square$

Exercise 3.16 Let P be a simplicial convex d-polytope and let Q be a simplicial convex d-polytope obtained by building a shallow pyramid over a single facet of P. Of course, this increases the number of vertices by one. Show that the h-vector of Q is obtained by increasing $h_i(P)$ by one, $i = 1, \ldots, d-1$. $\square$

Exercise 3.17 A simplicial convex d-polytope is called *stacked* if it can be obtained from a d-simplex by repeatedly building shallow pyramids over facets. What do the h-vector and the f-vector of a stacked d-polytope with n vertices look like? $\square$

Exercise 3.18 Let P be a d-polytope with n vertices such that $f_{j-1}(P) = \binom{n}{j}$, $j = 0, \ldots, \lfloor d/2 \rfloor$. Prove that $h_i(P) = \binom{n-d+i-1}{i}$, $i = 0, \ldots, \lfloor d/2 \rfloor$. Suggestion: Consider the lower powers of t in $f(P,t)$ and $h(P,t)$. $\square$

Exercise 3.19

1. Suppose P is a simplicial d-polytope and P' is a bipyramid over P. What is the relationship between $h(P)$ and $h(P')$?

2. Let P_1 be any 1-polytope (line segment), and let P_k be a bipyramid over P_{k-1}, $k = 2, 3, \ldots$. (Such P_k are combinatorially equivalent to *d-cross-polytopes*, which are dual to d-cubes.) Find formulas for $h(P_k)$ and $f(P_k)$.

$\square$

3.5 The Affine Span of $f(\mathcal{P}_s^d)$

For a simplicial d-polytope P, $d \geq 1$, consider the equation $h_i = h_{d-i}$. Obviously if $i = d-i$ then the equation is trivial, so let's assume that $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ (in particular, $d-i > i$). Then, as a linear combination of $f_{-1}, \ldots, f_{d-1}$, h_{d-i} contains the term f_{d-i-1}, whereas h_i does not. So the equation is nontrivial, $i = 0, \ldots, \lfloor (d-1)/2 \rfloor$. Clearly these equations form a linearly independent set of $\lfloor (d-1)/2 \rfloor + 1 = \lfloor (d+1)/2 \rfloor$ linear equations, so the dimension of the affine span of f-vectors $(f_0, \ldots, f_{d-1})$ of simplicial d-polytopes is at most $d - \lfloor (d+1)/2 \rfloor = \lfloor d/2 \rfloor$.

Let $m = \lfloor d/2 \rfloor$. To verify that the $\dim \operatorname{aff} f(\mathcal{P}_s^d) = m$, we need to find a collection of $m+1$ affinely independent f-vectors. (The notation aff denotes *affine span*.) Fortunately, there is a class of simplicial d-polytopes, called *cyclic polytopes*, which accomplishes this. We'll study cyclic polytopes a bit later, but for now it suffices to know that $f_{j-1} = \binom{n}{j}$, $j = 0, \ldots, m$, for cyclic d-polytopes $C(n,d)$ with n vertices.

Exercise 3.20 Prove that the set $\{f(C(n,d)) : n = d+1, \ldots, d+m+1\}$ is affinely independent. Suggestion: Write these vectors as rows of a matrix, throw away all but the first m columns, append an initial column of 1's, and then show that this matrix has full row rank by subtracting adjacent rows from each other. □

Theorem 3.21 *The dimension of* $\operatorname{aff} f(\mathcal{P}_s^d)$ *is* $\lfloor d/2 \rfloor$, *and* $\operatorname{aff} f(\mathcal{P}_s^d) = \{(f_0, \ldots, f_{d-1}) : h_i = h_{d-i},\ i = 0, \ldots, \lfloor (d-1)/2 \rfloor\}$.

The Dehn-Sommerville Equations can be expressed directly in terms of the f-vector. Here is one way (see [Grü67]):

Theorem 3.22 *If* $f \in f(\mathcal{P}_s^d)$ *then*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k,\ -1 \leq k \leq d-1.$$

The dual result for simple polytopes (see [Brø83]) is:

Theorem 3.23 *If* $f = (f_0, \ldots, f_d)$ *is the* f*-vector of a simple* d*-polytope, then*

$$\sum_{j=0}^{i} (-1)^j \binom{d-j}{d-i} f_j = f_i,\ i = 0, \ldots, d.$$

3.6 Vertex Figures

Let's return to a simplicial d-polytope P. Assume that v is a vertex of P, and Q is a vertex figure of P at v. Define B to be the collection of faces of P that do not contain v. It is a fact that Q is a simplicial $(d-1)$-polytope. In the following formulas we take $f_{-2}(Q) = h_{-1}(Q) = h_d(Q) = 0$.

Theorem 3.24 *Let P, Q, and B be as above. Then*

1. $f_j(P) = f_j(B) + f_{j-1}(Q)$, $j = -1, \ldots, d-1$.
2. $h_i(P) = h_i(B) + h_{i-1}(Q)$, $i = 0, \ldots, d$.
3. $h_i(Q) - h_{i-1}(Q) = h_i(B) - h_{d-i}(B)$, $i = 0, \ldots, d$.

PROOF.

1. This is clear because every j-face of P either does not contain v, in which case it is a j-face of B, or else does contain v, in which case it corresponds to a $(j-1)$-face of Q.

2. Expressing (1) in terms of polynomials, we get

$$f(P,t) = f(B,t) + tf(Q,t).$$

   So

$$(1-t)^d f(P, \frac{t}{1-t}) = (1-t)^d f(B, \frac{1}{1-t}) + (1-t)^d \frac{t}{1-t} f(Q, \frac{t}{1-t}),$$
$$h(P,t) = h(B,t) + th(Q,t),$$

   and equating coefficients of t^i gives (2).

3. The Dehn-Sommerville Equations for P and Q are equivalent to the statements

$$h(P,t) = t^d h(P, \frac{1}{t})$$

   and

$$h(Q,t) = t^{d-1} h(Q, \frac{1}{t}).$$

   Therefore

$$\begin{aligned} h(B,t) - t^d h(B, \tfrac{1}{t}) &= h(P,t) - th(Q,t) - t^d h(P, \tfrac{1}{t}) + t^d \tfrac{1}{t} h(Q, \tfrac{1}{t}) \\ &= t^{d-1} h(Q, \tfrac{1}{t}) - th(Q,t) \\ &= h(Q,t) - th(Q,t). \end{aligned}$$

   Equating coefficients of t^i gives (3). □

This theorem tells us that the h-vectors, and hence the f-vectors, of both P and Q, are completely determined by the h-vector, and hence the f-vector, of B. We can use (3) to iteratively compute $h_0(Q), h_1(Q), h_2(Q), \ldots$, and then determine $h(P)$ from (2).

If we think of the boundary complex of P as a hollow $(d-1)$-dimensional "simplicial sphere", then B is a $(d-1)$-dimensional "simplicial ball", and the faces on the "boundary" of B correspond to the faces of Q. Actually (though I haven't defined the terms), the Dehn-Sommerville Equations apply to any simplicial sphere, so this theorem can be generalized to prove that the f-vector of the boundary of any simplicial ball is completely determined by the f-vector of the ball itself.

Example 3.25 Suppose P is a simplicial 7-polytope, v is a vertex of P, and B is defined as above. Assume that

$$f(B) = (11, 55, 165, 314, 365, 234, 63).$$

Let's find $f(P)$ and $f(Q)$.

$$f(B,t) = 1 + 11t + 55t^2 + 165t^3 + 314t^4 + 365t^5 + 234t^6 + 63t^7,$$

$$\begin{aligned} h(B,t) &= (1-t)^7 f(B, \frac{t}{1-t}) \\ &= 1 + 4t + 10t^2 + 20t^3 + 19t^4 + 7t^5 + 2t^6. \end{aligned}$$

Set up an "addition" for $h(B)$, $h(Q)$, and $h(P)$:

$$\begin{array}{c|cccccccc} h(B) & 1 & 4 & 10 & 20 & 19 & 7 & 2 & 0 \\ h(Q) & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline h(P) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

The missing two rows must each be symmetric, which forces the solution

$$\begin{array}{c|cccccccc} h(B) & 1 & 4 & 10 & 20 & 19 & 7 & 2 & 0 \\ h(Q) & + & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\ \hline h(P) & 1 & 5 & 13 & 26 & 26 & 13 & 5 & 1 \end{array}$$

So

$$\begin{aligned} h(Q,t) &= 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6, \\ h(P,t) &= 1 + 5t + 13t^2 + 26t^3 + 26t^4 + 13t^5 + 5t^6 + t^7. \end{aligned}$$

Using $f(Q,t) = (1+t)^6 h(\frac{t}{1+t})$ and $f(P,t) = (1+t)^7 h(\frac{t}{1+t})$, we compute

$$\begin{aligned} f(Q) &= (9, 36, 81, 108, 81, 27), \\ f(P) &= (12, 64, 201, 395, 473, 315, 90). \end{aligned}$$

□

Exercise 3.26 Let B be as in the previous theorem and let $\operatorname{int} B$ be the set of faces of B that do not correspond to faces of Q (are not on the "boundary" of B). Prove that $h(\operatorname{int} B) = (h_d(B), \ldots, h_0(B))$. $\square$

Messing around a bit, we can come up with an explicit formula for $f(Q)$ in terms of $f(B)$ (we omit the proof):

Theorem 3.27 *Assume that B and Q are as in the previous theorem. Then*

$$f_k(Q) = f_k(B) + (-1)^d \sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(B), \quad -1 \le k \le d-2.$$

Comparing this to Theorem 3.22, we see that $f(Q)$ measures the deviation of $f(B)$ from satisfying the Dehn-Sommerville Equations.

It is a fact that every unbounded simple d-polyhedron R is dual to a certain B as occurs above, and that the collection of unbounded faces of R is dual to Q. So the previous theorem can be used to get an explicit formula for the number of unbounded faces of R in terms of the f-vector of R.

Theorem 3.28 *If $f = (f_0, \ldots, f_d)$ is the f-vector of a simple d-polyhedron R, and if f_j^u is the number of unbounded j-faces of R, $j = 1, \ldots, d$, then*

$$f_i^u(R) = f_i - \sum_{j=0}^{i} (-1)^j \binom{d-j}{d-i} f_j, \; i = 0, \ldots, d.$$

3.7 Notes

In 1905 Dehn [Deh05] worked on the equations for $d = 4$ and $d = 5$ and conjectured the existence of analogous equations for $d > 5$. Sommerville [Som27] derived the complete system of equations for arbitrary d in 1927. Klee [Kle64] in 1964 rediscovered the Dehn-Sommerville Equations in the more general setting of manifolds and incidence systems. In addition to d-polytopes, the equations hold also for simplicial $(d-1)$-spheres, triangulations of homology $(d-1)$-spheres, and Klee's Eulerian $(d-1)$-spheres. See [Grü67] for more historical details and generalizations. McMullen and Walkup [MW71] (see also [MS71]) introduced the important notion of the h-vector (though they used the letter g). (I have heard, however, that Sommerville may have also formulated the Dehn-Sommerville Equations in a form equivalent to $h_i = h_{d-i}$—I have yet to check this.) Stanley [Sta75a, Sta75b, Sta77, Sta78, Sta80, Sta96] made the crucial connections between the h-vector and algebra, some of which we shall discuss later.

For more on the Dehn-Sommerville Equations, see [Brø83, Grü67, MS71, Zie95].

4 Shellability

We mentioned that early "proofs" of Euler's Relation assumed that (the boundaries of) polytopes are shellable. But this wasn't established until 1970 by Bruggesser and Mani [BM71]—with a wonderful insight reaffirming that hindsight is 20/20.

Recall for any d-polytope P that $\mathcal{F}(P)$ denotes the set of all faces of P, both proper and improper, and $\mathcal{F}(\operatorname{bd} P) = \mathcal{F}(P) \setminus \{P\}$. Suppose S is a nonempty subset of $\mathcal{F}(\operatorname{bd} P)$ with the property that if F and G are two faces of P with $F \subseteq G \in S$, then $F \in S$; that is to say, the collection S is *closed under inclusion*.

We define S to be *shellable* if the following conditions hold:

1. For every j-face F in S there is a facet of P in S containing F (S is *pure*).

2. The facets in S can be ordered $F_1, \ldots, F_n$ such that for every $k = 2, \ldots, n$, $T_k := \mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ is a shellable collection of faces of the $(d-1)$-polytope F_k.

Such an ordering of the facets in S is called a *shelling order*. We say that a d-polytope P is *shellable* if $\mathcal{F}(\operatorname{bd} P)$ is shellable.

We note first that if S consists of a single facet F of P and all of the faces contained in F, then condition (1) is trivially true and condition (2) is vacuously true. So every 0-polytope is shellable (there is only one facet—the empty set). It is easy to check that every 1-polytope is shellable (try it). Condition (2) implies in particular that the intersection $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ is nonempty, since the empty set is a member of $\mathcal{F}(F_1), \ldots, \mathcal{F}(F_k)$.

Exercise 4.1

1. Let P be a 2-polytope. Use the definition to characterize when a set S of faces of P is shellable. $\square$

2. Investigate the analogous question when P is a 3-polytope.

Exercise 4.2 Let P be a d-simplex. Then P has $d+1$ vertices, and every subset of vertices determines a face of P. Let $\{F_1, \ldots, F_m\}$ be any subset of facets of P. Prove that $S := \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_m)$ is shellable, and the facets of S can be shelled in any order. $\square$

Theorem 4.3 (Bruggesser-Mani 1971) *Let P be a d-polytope. Then P is shellable. Further, there is a shelling order $F_1, \ldots, F_n$ of the facets of P such that for every $k = 1, \ldots, n$, there is a shelling order $G_1^k, \ldots, G_{n_k}^k$ of the facets of F_k for which $\mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ equals $\mathcal{F}(G_1^k) \cup \cdots \cup \mathcal{F}(G_\ell^k)$ for some $0 \leq \ell \leq n_k$. Moreover, $\ell = n_k$ if and only if $k = n$.*

The case $\ell = 0$ occurs if and only if $k = 1$, and is just a sneaky way of saying that $\mathcal{F}(\operatorname{bd} F_1)$ is shellable.

Imagine that P is a planet and that you are in a rocket resting on one of the facets of P. Take off from the planet in a straight line, and create a list of the facets of P in the order in which they become "visible" to you (initially one one facet is visible). Proceed "to infinity and beyond," returning toward the planet along the same line but from the opposite direction. Continue adding facets to your list, but this time in the order in which they disappear from view. The last facet on the list is the one you land on. Bruggesser and Mani proved that this is a shelling order (though I believe they traveled by balloon instead of by rocket). The proof given here is a dual proof.

PROOF. We'll prove this by induction on d. It's easy to see the result is true if $d = 0$ and $d = 1$, so assume $d > 1$. Let $P^* \subset \mathbf{R}^d$ be a polytope dual to P. Choose a vector $c \in \mathbf{R}^d$ such that $c^T v$ is different for each vertex v of P^*. Order the vertices $v_1, \ldots, v_n$ by increasing value of $c^T v_i$. The vertices of P^* correspond to the facets $F_1, \ldots, F_n$ of P. We claim that that this is a shelling order.

For each $i = 1, \ldots, n$, define $\mathcal{F}^*(v_i)$ to be the set of faces of P^* that contain v_i (including P^* itself). Let $S_k(P^*) = \mathcal{F}^*(v_1) \cup \cdots \cup \mathcal{F}^*(v_k)$. We will prove that $S_k(P^*)$ is dual (anti-isomorphic) to a shellable collection of faces of P, $k = 1, \ldots, n$.

The result follows from the following observations about the duality between P and P^*:

1. As mentioned above, the facets $F_1, \ldots, F_n$ of P correspond to the vertices $v_1, \ldots, v_n$ of P^*.

2. For each k, $\mathcal{F}(F_k)$ is dual to the set $\mathcal{F}^*(v_k)$.

3. For each k, the facets $G_1^k, \ldots, G_{n_k}^k$ of F_k correspond to the edges of P^* that contain v_k, which in turn correspond to the vertices $v_1^k, \ldots, v_{n_k}^k$ of a vertex figure F_k^* of P^* at v_k. The facets $G_1^k, \ldots, G_{n_k}^k$ are to be ordered by the induced ordering of $v_1^k, \ldots, v_{n_k}^k$ by c. (In constructing the vertex figure, be sure that its vertices have different values of $c^T v_i^k$.)

4. For each k and i, the set $\mathcal{F}(G_i^k)$ is dual to the set $\mathcal{F}^*(v_i^k)$, defined to be the set of faces of F_k^* that contain v_i^k.

5. For each k, $\mathcal{F}^*(v_k) \cap (\mathcal{F}^*(v_1) \cup \cdots \cup \mathcal{F}^*(v_{k-1}))$ is dual to the collection of faces in T_k. It consists of all of the faces of P^* containing both v_k and some "lower" vertex v_i, $i = 1, \ldots, k-1$. Equivalently, these are the faces of P^* containing at least one edge joining v_k to some lower vertex v_i, $i = 1, \ldots, k-1$. Thus, looking at the vertex figure

F_k^*, this set of faces corresponds to the set $S_\ell(F_k^*) = \mathcal{F}^*(v_1^k) \cup \cdots \cup \mathcal{F}^*(v_\ell^k)$ for some $\ell \le n_k$. This set $S\ell(F_k^*)$ is dual to a shellable collection of faces of F_k by induction. Further, $\ell = n_k$ if and only if $k = n$. □

Exercise 4.4 (Line Shellings) Assume $P \subset \mathbf{R}^d$ is a d-polytope containing the origin O in its interior. Let $F_1, \ldots, F_n$ be the facets of P and let $H_1, \ldots, H_n$ be the respective supporting hyperplanes for these facets. Choose a direction $c \in \mathbf{R}^d$ and define the line $L := \{tc : t \in \mathbf{R}\}$. Assume that c has been chosen such that as you move along L you intersect the various H_i one at a time (why does such a line exist?). By relabeling, if necessary, assume that as you start from O and move in one direction along L (t positive and increasing) you encounter the H_i in the order $H_1, \ldots, H_\ell$. Now move toward O from infinity along the other half of L (t negative and increasing) and assume that you encounter the remaining H_i in the order $H_{\ell+1}, \ldots, H_n$. Prove that $F_1, \ldots, F_n$ constitutes a shelling order by examining the polar dual P^*. (Such shellings are called *line shellings*.) □

Exercise 4.5 Find a 2-polytope P and some ordering of the facets of P that is a shelling, but not a line shelling, regardless of the location of O. □

Exercise 4.6 Let P be a d-polytope and $F_1, \ldots, F_n$ be a line shelling order of its facets. For $k = 1, \ldots, n$, let $S_k = \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_k)$. For any subset of faces S of P define

$$\hat{\chi}(S) := \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j(S).$$

Prove Euler's Relation by showing that

$$\hat{\chi}(S_k) = \begin{cases} 0, & k = 1, \ldots, n-1, \\ 1, & k = n. \end{cases}$$

□

Exercise 4.7 Let P be a d-polytope. If $F_1, \ldots, F_n$ is a line-shelling of P, then the only time $F_k \cap (F_1 \cup \cdots \cup F_{k-1})$ contains *all* of the facets of F_k is when $k = n$. Show that the same is true for arbitrary shelling orders, not just line shellings. Suggestion: Use Exercise 4.6. □

Exercise 4.8 Let P be a d-polytope and v be any vertex of P. Prove that there is a shelling order of P such that the set of facets containing v are shelled first. □

Exercise 4.9 Let P be a simplicial d-polytope. Explain how the h-vector of P can be calculated from a shelling order of its facets. Do this in the following way: Assume that $F_1, \ldots, F_n$ is a shelling order of the facets of a simplicial d-polytope. Prove that for every $k = 1, \ldots, n$ there is a face G_k in $\mathcal{F}(F_k)$ such that $\mathcal{F}(F_k) \cap (\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1}))$ is the set of all faces of F_k not containing G_k. Then show that

$$h_i(S_k) = \begin{cases} h_i(S_{k-1}) + 1, & i = f_0(G_k), \\ h_i(S_{k-1}), & \text{otherwise.} \end{cases}$$

□

Exercise 4.10 Finish the proof of Gram's Theorem (Section 2.4) by showing that

$$\sum_{F:0\leq\dim F\leq d} (-1)^{\dim F} a_F(x) = 0$$

if $x \notin P$. Suggestion: Suppose P is a d-polytope in $\mathbf{R}^d$. Each facet F_i has a unique supporting hyperplane H_i. Let H_i^+ be the closed halfspace associated with H_i containing P, and H_i^- be the opposite closed halfspace. Let F be any proper face of P. Define x to be *beyond* F (or F to be *visible* from x) if and only if there is at least one i such that $F \subset F_i$ and $x \in H_i^- \setminus H_i$. Note that $a_F(x) = 0$ if and only if F is visible from x. Now prove that the set of faces visible from x is shellable. Apply Euler's Relation (Exercise 4.6). □

More details on shellings can be found in [Brø83, Grü67, MS71, Zie95]. Ziegler [Zie95] proves that not all 4-polytopes are *extendably shellable*. In particular, there exists a 4-polytope P and a collection $F_1, \ldots, F_m$ of facets of P such that $\mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_m)$ is shellable with shelling order $F_1, \ldots, F_m$, but this cannot be extended to a shelling order $F_1, \ldots, F_m, F_{m+1}, \ldots, F_n$ of all of the facets of P.

5 The Upper Bound Theorem

At about the same time that polytopes were proved to be shellable, two important extremal f-vector results were also settled. What are the maximum and minimum values of $f_j(P)$ for simplicial d-polytopes with n vertices? We now know that the maxima are attained by cyclic polytopes, which we discuss in this section, and the minima by stacked polytopes, which we tackle in the next.

A good reference for this section is [MS71]. In fact, McMullen discovered the proof of the Upper Bound Theorem while writing this book with Shephard. Originally the intent was to report on progress in trying to solve what was then known as the Upper Bound Conjecture. See also [Brø83, Zie95].

To construct cyclic polytopes, consider the *moment curve* $\{m(t) := (t, t^2, \dots, t^d) : t \in \mathbf{R}\} \subset \mathbf{R}^d$ and choose $n \geq d+1$ distinct points $v_i = m(t_i)$ on the curve, $t_1 < \cdots < t_n$. Let $V = \{v_1, \dots, v_n\}$ and set $C(n,d) := \operatorname{conv} V$, the convex hull of V. This is called a *cyclic polytope.*

First we will show that $C(n,d)$ is a simplicial d-polytope. Let W be the set of any d points on the moment curve and let $a_1x_1 + \cdots + a_dx_d = a_0$ be the equation of any hyperplane containing W. Then $a_1t_i + a_2t_i^2 + \cdots + a_dt_i^d = a_0$ if $v_i = m(t_i) \in W$. Therefore the polynomial $a_1t + a_2t^2 + \cdots + a_dt^d - a_0$ has at least d roots. But being nontrivial and of degree $\leq d$ it has at most d roots. Therefore there can be no other points of the moment curve on H besides W. We conclude that $C(n,d)$ is full-dimensional, and every facet contains only d vertices and hence is a simplex.

Now we prove that $C(n,d)$ has a remarkable number of lower dimensional faces.

Theorem 5.1 *Let $W \subset V$ have cardinality at most $\lfloor d/2 \rfloor$. Then* $\operatorname{conv} W$ *is a face of* $C(n,d)$*. Consequently* $f_{j-1}(C(n,d)) = \binom{n}{j}$, $j = 0, \dots, \lfloor d/2 \rfloor$.

Proof. Consider the polynomial

$$p(t) = \prod_{v_i \in W} (t - t_i)^2.$$

It has degree at most d, so it can be written $a_0 + a_1t + \cdots + a_dt^d$. Note that t_i is a root if $v_i \in W$ and that $p(t_i) > 0$ if $v_i \in V \setminus W$. So the vertices of V which lie on the hyperplane H whose equation is $a_1x_1 + \cdots + a_dx_d = -a_0$ are precisely the vertices in W, and H is a supporting hyperplane to $C(n,d)$. So we have a supporting hyperplane for $\operatorname{conv} W$, which is therefore a face of $C(n,d)$. $\square$

The cyclic polytope $C(n,d)$ obviously has the maximum possible number of j-faces, $1 \le j \le \lfloor d/2 \rfloor - 1$, of any d-polytope with n vertices. Polytopes of dimension d for which $f_j = \binom{n}{j+1}$, $j = 0, \ldots, \lfloor d/2 \rfloor - 1$, are called *neighborly*. It might be supposed that $C(n,d)$ has the maximum possible number of higher dimensional faces as well. Motzkin [Mot57] implicitly conjectured this, and McMullen[McM70] proved this to be the case.

We first examine the h-vector. Note that $f(C(n,d),t)$ agrees with $(1+t)^n$ in the coefficients of t^i, $i = 0, \ldots, \lfloor d/2 \rfloor$. Knowing that h_i depends only upon $f_{-1}, \ldots, f_{i-1}$ (see Equation (4)), we have that $h(C(n,d),t) = (1-t)^d f(C(n,d), \frac{t}{1-t})$ agrees with

$$(1-t)^d (1 + \frac{t}{1-t})^n = (1-t)^{d-n} = (1 + t + t^2 + \cdots)^{n-d}$$

in the coefficients of t^i, $i = 0, \ldots, \lfloor d/2 \rfloor$. Therefore,

$$h_i(C(n,d)) = \binom{n-d+i-1}{i}, \ i = 0, \ldots, \lfloor d/2 \rfloor$$

(verify this!). The second half of the h-vector is determined by the Dehn-Sommerville Equations.

Theorem 5.2 (Upper Bound Theorem, McMullen 1970) *If P is a convex d-polytope with n vertices, then $f_j(P) \le f_j(C(n,d))$, $j = 1, \ldots, d-1$.*

PROOF. Perturb the vertices of P slightly, if necessary, so that we can assume P is simplicial. This will not decrease any component of the f-vector and will not change the number of vertices. Since the components of the h-vector are nonnegative combinations of the components of the f-vector (Equation (5)), it suffices to show that $h_i(P) \le h_i(C(n,d))$ for all i. Because of the Dehn-Sommerville Equations, it is enough to prove $h_i(P) \le \binom{n-d+i-1}{i}$, $i = 1, \ldots, \lfloor d/2 \rfloor$.

Choose any simple d-polytope $Q \subset \mathbf{R}^d$ dual to P and recall that $h_i(P)$ by definition equals $h_i(Q)$, which equals the number of vertices of Q of indegree i whenever the edges are oriented by any sufficiently general vector $c \in \mathbf{R}^d$. Let F be any facet of Q. Then $h(F)$ can be obtained using the same vector c by simply restricting attention to the edges of Q in F.

Claim 1. $\sum_F h_i(F) = (i+1)h_{i+1}(Q) + (d-i)h_i(Q)$. Let v be any vertex of Q of indegree $i+1$. We can drop any one of the $i+1$ edges entering v and find the unique facet F containing the remaining $d-1$ edges incident to v. The vertex v will have indegree i when restricted to F. On the other hand, let v be any vertex of Q of indegree i. We can drop any one of the $d-i$ edges leaving v and find the unique facet F containing the remaining $d-1$ edges incident to v. This time the vertex v will have indegree $i+1$ when restricted to F. These two cases account for all vertices of indegree i in the sum $\sum_F h_i(F)$.

Claim 2. $\sum_F h_i(F) \leq n h_i(Q)$. For, consider any facet F. We may choose a vector c so that the ordering of vertices of Q by c begins with the vertices of F (choose c to be a slight perturbation of an inner normal vector of F). It is now easy to see that with this ordering, a contribution to $h_i(F)$ gives rise to a contribution to $h_i(Q)$. Thus $h_i(F) \leq h_i(Q)$, and summing over the facets of Q proves the claim.

From the two claims we can easily prove $(i+1)h_{i+1}(Q) \leq (n-d+i)h_i(Q)$ from which $h_i(Q) \leq \binom{n-d+i-1}{i}$ follows quickly by induction on i. □

What can be said about a d-polytope P for which $f(P) = f(C(n,d))$? Obviously P must be simplicial and neighborly. Shemer [She82] has shown that there are very many non-cyclic neighborly polytopes. If d is even, every neighborly d-polytope is simplicial, but nonsimplicial neighborly d-polytopes exist when d is odd [Brø83, Grü67].

Again, assume that $V = \{v_i = m(t_i),\ i = 1, \ldots, n\}$, $t_1 < \cdots < t_n$, $n \geq d+1$, is the set of vertices of a cyclic polytope. Suppose W is a subset of V of cardinality d. When does W correspond to a facet of $C(n,d) = \operatorname{conv} V$? Looking back at the discussion at the beginning of this section, in which we proved $C(n,d)$ is simplicial, we observe that the polynomial

$$p(t) = \prod_{v_i \in W} (t - t_i)$$

changes sign at each of its roots, and that W corresponds to a facet if and only if the numbers $p(t_i)$ all have the same sign for $v_i \notin W$. Therefore, "between" every two nonelements of W must lie an even number of elements of W. The next theorem is immediate (see [Brø83, Grü67, MS71, Zie95]).

Theorem 5.3 (Gale's Evenness Condition) *The subset W corresponds to a facet of $C(n,d)$ if and only if for every pair $v_k, v_\ell \notin W$, $k < \ell$, the set $W \cap \{v_i : k < i < \ell\}$ has even cardinality.*

The above theorem shows that the combinatorial structure of the cyclic polytope does not depend upon the particular choice of the values t_i, $i = 1, \ldots, n$. Thus, from a combinatorial point of view, we are justified in calling $C(n,d)$ *the* cyclic d-polytope with n vertices.

Example 5.4 Here is a representation of the facets of $C(8,5)$:

1	2	3	4	5	6	7	8
1	2	3	4	5			
1	2	3		5	6		
1		3	4	5	6		
1	2	3			6	7	
1		3	4		6	7	
1			4	5	6	7	
1	2	3				7	8
1		3	4			7	8
1			4	5		7	8
1				5	6	7	8
1	2	3	4				8
1	2		4	5			8
	2	3	4	5			8
1	2			5	6		8
	2	3		5	6		8
		3	4	5	6		8
1	2				6	7	8
	2	3			6	7	8
		3	4		6	7	8
			4	5	6	7	8

□

Exercise 5.5 (Ziegler [Zie95]) Show (bijectively) that the number of ways in which $2k$ elements can be chosen from $\{1,\ldots,n\}$ in "even blocks of adjacent elements" is $\binom{n-k}{k}$. Thus, derive from Gale's evenness condition that the formula for the number of facets of $C(n,d)$ is

$$f_{d-1}(C(n,d)) = \binom{n-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor} + \binom{n-1-\lceil\frac{d-1}{2}\rceil}{\lfloor\frac{d-1}{2}\rfloor},$$

where $\lceil\cdot\rceil$ is the *least integer function*, with $\lceil\frac{k}{2}\rceil = k - \lfloor k/2\rfloor$. Here the first term corresponds to the facets for which the first block is even, and the second term corresponds to the cases where the first block is odd. Deduce

$$f_{d-1}(C(n,d)) = \begin{cases} \frac{n}{n-k}\binom{n-k}{k}, & \text{for } d=2k \text{ even,} \\ \\ 2\binom{n-k-1}{k}, & \text{for } d=2k+1 \text{ odd.} \end{cases}$$

□

As a consequence, for fixed d the number of facets of $C(n, d)$ grows like a polynomial of degree $\lfloor d/2 \rfloor$ (see [Zie95]). Dually, this gives us an upper bound on the number of basic feasible solutions of a linear program in d variables described by n linear inequalities.

Exercise 5.6 (Ziegler [Zie95]) Show that if a polytope is k-*neighborly* (every subset of vertices of cardinality at most k corresponds to a face), then every $(2k-1)$-face is a simplex. Conclude that if a d-polytope is $(\lfloor d/2 \rfloor + 1)$-neighborly, then it is a simplex. □

6 The Lower Bound Theorem

6.1 Stacked Polytopes

What is the least number of j-faces that a simplicial d-polytope with n vertices can have? The answer turns out to be achieved simultaneously for all j by a stacked polytope. A d-polytope $P(n,d)$ is *stacked* if either $n = d+1$ and it is a d-simplex, or else $n > d+1$ and $P(n,d)$ is obtained by building a shallow pyramid over one of the facets of some stacked polytope $P(n-1,d)$. Unlike the cyclic polytopes, not all stacked d-polytopes with n vertices are combinatorially equivalent. See [Brø83, Grü67].

To calculate the f-vector of a stacked polytope $P = P(n,d)$, note first that for $n > d+1$,

$$f_j(P(n,d)) = f_j(P(n-1,d)) + \begin{cases} \binom{d}{j}, & j = 0,\dots,d-2, \\ \\ d-1, & j = d-1. \end{cases}$$

The next exercise shows that the h-vector has a particularly simple form.

Exercise 6.1 Prove

$$f_j(P(n,d)) = \begin{cases} \binom{d+1}{j+1} + (n-d-1)\binom{d}{j}, & j = 0,\dots,d-2, \\ \\ (d+1) + (n-d-1)(d-1), & j = d-1, \end{cases}$$

and

$$h_i(P(n,d)) = \begin{cases} 1, & i = 0 \text{ or } i = d, \\ n-d, & i = 1,\dots,d-1. \end{cases}$$

□

Theorem 6.2 (Lower Bound Theorem, Barnette 1971, 1973) *If P is a simplicial convex d-polytope with n vertices, then $f_j(P) \geq f_j(P(n,d))$, $j = 1,\dots,d-1$.*

The lower bound for $d = 4$ was stated by Brückner [Brü09] in 1909 as a theorem, but his proof was later shown to be invalid (Steinitz [Ste22]). Barnette [Bar71] first proved the case $j = d-1$, and then the remaining cases [Bar73]. His proof is reproduced in [Brø83]. He also proved that if $d \geq 4$ and $f_{d-1}(P) = f_{d-1}(P(n,d))$, then P is a stacked d-polytope. Billera and Lee [BL81] extended this by showing that if $d \geq 4$ and $f_j(P) = f_j(P(n,d))$ for any single value of j, $j = 1,\dots,d-1$, then P is stacked.

Even though Barnette's proof is not difficult, we will present the later proof by Kalai [Kal87], which provides some deep insight into connections between the h-vector and other properties of simplicial polytopes.

6.2 Motion and Rigidity

The vertices and edges of a convex d-polytope provide an example of a framework. More generally, a *(bar and joint) framework* G in $\mathbf{R}^d$ is a finite collection of vertices (*joints*) $v_i \in \mathbf{R}^d$, $i \in V := \{1, \ldots, n\}$, and edges (*bars*) $v_i v_j := \operatorname{conv}\{v_i, v_j\}$, $i \neq j$, $ij := (i,j) \in E \subset V \times V$. (We assume $ij \in E$ if and only if $ji \in E$.) We do not care whether the vertices are all distinct, or whether the edges coincidentally intersect each other at other than their common endpoints. Define the *dimension* of the framework to be $\dim \operatorname{aff}\{v_1, \ldots, v_n\}$.

Now let $I \subseteq \mathbf{R}$ be an open interval and parameterize the vertices as $v_i(t)$ such that $v_i = v_i(0)$, $i = 1, \ldots, n$, and $\|v_i(t) - v_j(t)\|^2 = \|v_i - v_j\|^2$, $ij \in E$, for all $t \in I$; i.e., no edge is changing length. This defines a *motion* of the framework. A motion of any framework can be induced by a Euclidean motion (such as a translation or rotation) of the entire space $\mathbf{R}^d$—such induced motions are called *trivial*. A framework admitting only trivial motions is *rigid*.

Exercise 6.3 Give some examples of motions of two-dimensional frameworks. □

Since each edge-length is constant during a motion, we have

$$\begin{aligned} 0 &= \tfrac{d}{dt}[(v_i(t) - v_j(t))^T(v_i(t) - v_j(t))] \\ &= 2(v_i(t) - v_j(t))^T(v_i'(t) - v_j'(t)),\ ij \in E. \end{aligned}$$

Setting $t = 0$ and $u_i := v_i'(0)$, $i = 1, \ldots, n$, we have

$$(v_i - v_j)^T(u_i - u_j) = 0 \text{ for all } ij \in E. \tag{6}$$

By definition, any set of vectors $u_1, \ldots, u_n \in \mathbf{R}^d$ that satisfies Equation (6) is said to be an *infinitesimal motion* of the framework. It can be checked that not every infinitesimal motion is derived from a motion. The set of infinitesimal motions of a framework is a vector space and is called the *motion space* of the framework.

Exercise 6.4 Give some examples of infinitesimal motions of two-dimensional frameworks. Find some that come from motions, and some that do not. □

Exercise 6.5 Prove that $u_1, \ldots, u_n$ is an infinitesimal motion if and only if the projections of the vectors u_i and u_j onto the vector $v_i - v_j$ agree for every $ij \in E$. □

When we have an infinitesimal motion of the vertices and edges of a polytope P, then we simply say we have an *infinitesimal motion of P*.

Some infinitesimal motions of a framework are clearly trivial—for example, we may choose the vectors u_i to be all the same. To make the notion of trivial more precise, we first define an *infinitesimal motion of* $\mathbf{R}^d$ to be an assignment of a vector $u \in \mathbf{R}^d$ to *every* point $v \in \mathbf{R}^d$ (u depends upon v) such that $(v - \overline{v})^T(u - \overline{u}) = 0$ for *every* pair of points $v, \overline{v} \in \mathbf{R}^d$. An infinitesimal motion of a framework is *trivial* if it is the restriction of some infinitesimal motion of $\mathbf{R}^d$ to that framework. A framework that admits only trivial infinitesimal motions is said to be *infinitesimally rigid*. Infinitesimal rigidity implies rigidity, but a framework can be rigid without being infinitesimally rigid. But in real life, I would rather rely upon scaffolding that is infinitesimally rigid!

Exercise 6.6 Give some examples of trivial and nontrivial infinitesimal motions of two-dimensional frameworks. □

Theorem 6.7 *Let $P \subset \mathbf{R}^d$ be a d-simplex with vertices $v_1, \ldots, v_{d+1}$, and $u_1, \ldots, u_{d+1}$ be an infinitesimal motion of P. Then u_{d+1} is determined by $u_1, \ldots, u_d$.*

Proof. Let $e_i = v_i - v_{d+1}$, $i = 1, \ldots, d$. These vectors are linearly independent. The projections of u_{d+1} and u_i onto e_i must agree, $i = 1, \ldots, d$, and u_{d+1} is determined by these d projections. □

Theorem 6.8 *The dimension of the motion space of a d-simplex $P \subset \mathbf{R}^d$ is $\binom{d+1}{2}$.*

Proof. Let P have vertices $v_1, \ldots, v_{d+1} \in \mathbf{R}^d$. Choose any vector $u_1 \in \mathbf{R}^d$. There are d degrees of freedom in this choice—one for each coordinate. Choose any vector $u_2 \in \mathbf{R}^d$ such that the projections p_1^2 of u_2 and of u_1 on the vector $v_1 - v_2$ agree. There are $d-1$ degrees of freedom in this choice, since you can freely choose the component of u_2 orthogonal to p_1^2. In general, for $k = 2, \ldots, d$, choose any vector $u_k \in \mathbf{R}^d$ such that the projections p_i^k of u_k and of u_i on the vectors $v_i - v_k$ agree, $i = 1, \ldots, k-1$. There are $d-k+1$ degrees of freedom in this choice, since you can freely choose the component of u_k orthogonal to the span of $p_1^k, \ldots, p_{k-1}^k$. The resulting set of vectors $u_1, \ldots, u_{d+1}$ is an infinitesimal motion of P, all infinitesimal motions of P can be constructed in this way, and there are $d+(d-1)+\cdots+2+1+0 = \binom{d+1}{2}$ degrees of freedom in constructing such a set of vectors. □

Theorem 6.9 *Let $P \subset \mathbf{R}^d$ be a d-simplex. Then P is infinitesimally rigid, and in fact every infinitesimal motion of P uniquely extends to an infinitesimal motion of $\mathbf{R}^d$.*

Proof. That the extension must be unique if it exists is a consequence of Theorem 6.7, for if v is any point in $\mathbf{R}^d$ and u is its associated infinitesimal motion vector, then u is uniquely determined by any subset of d vertices of P whose affine span misses v.

To show that an extension is always possible, let $v_1, \ldots, v_{d+1}$ be the vertices of P, and $u_1, \ldots, u_{d+1}$ be an infinitesimal motion of P. Then $(v_i - v_j)^T(u_i - u_j) = 0$ for all i, j. So

$$v_i^T u_i + v_j^T u_j = v_i^T u_j + v_j^T u_i \tag{7}$$

for all i, j.

Any $v \in \mathbf{R}^d$ can be uniquely written as an affine combination of $v_1, \ldots, v_{d+1}$:

$$v = \sum_{i=1}^{d+1} a_i v_i,$$

where $\sum_{i=1}^{d+1} a_i = 1$. Define

$$u = \sum_{i=1}^{d+1} a_i u_i.$$

We claim that this defines an infinitesimal motion of $\mathbf{R}^d$.

Choose $v, \overline{v} \in \mathbf{R}^d$. Assume that

$$v = \sum_{i=1}^{d+1} a_i v_i,$$

$$\overline{v} = \sum_{i=1}^{d+1} b_i v_i,$$

where $\sum_{i=1}^{d+1} a_i = \sum_{i=1}^{d+1} b_i = 1$. Let

$$u = \sum_{i=1}^{d+1} a_i u_i,$$

$$\overline{u} = \sum_{i=1}^{d+1} b_i u_i.$$

We must show that $(v - \overline{v})^T(u - \overline{u}) = 0$; i.e.,

$$\left(\sum_i a_i v_i - \sum_i b_i v_i\right)^T \left(\sum_j a_j u_j - \sum_j b_j u_j\right) = 0.$$

Equivalently, we must show

$$\sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j b_i b_j v_i^T u_j = \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j b_i a_j v_i^T u_j. \tag{8}$$

Now I know there must be slicker way of doing this, but here is one way. Multiply Equation (7) by $a_i a_j$ and sum over i and j:

$$\begin{aligned}
\sum_i \sum_j a_i a_j v_i^T u_i + \sum_i \sum_j a_i a_j v_j^T u_j &= \sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j a_i a_j v_j^T u_i \\
\sum_j a_j \sum_i a_i v_i^T u_i + \sum_i a_i \sum_j a_j v_j^T u_j &= \sum_i \sum_j a_i a_j v_i^T u_j + \sum_i \sum_j a_i a_j v_i^T u_j \\
\sum_i a_i v_i^T u_i + \sum_j a_j v_j^T u_j &= 2 \sum_i \sum_j a_i a_j v_i^T u_j \\
2 \sum_i a_i v_i^T u_i &= 2 \sum_i \sum_j a_i a_j v_i^T u_j \\
\sum_i a_i v_i^T u_i &= \sum_i \sum_j a_i a_j v_i^T u_j.
\end{aligned}$$

Similarly,

$$\sum_i b_i v_i^T u_i = \sum_i \sum_j b_i b_j v_i^T u_j.$$

Therefore, the left-hand side of Equation (8) equals

$$\sum_i a_i v_i^T u_i + \sum_i b_i v_i^T u_i. \tag{9}$$

Now multiply Equation (7) by $a_i b_j$ and sum over i and j:

$$\begin{aligned}
\sum_i \sum_j a_i b_j v_i^T u_i + \sum_i \sum_j a_i b_j v_j^T u_j &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_i b_j v_j^T u_i \\
\sum_i a_i v_i^T u_i + \sum_j b_j v_j^T u_j &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_j b_i v_i^T u_j \\
\sum_i a_i v_i^T u_i + \sum_i b_i v_i^T u_i &= \sum_i \sum_j a_i b_j v_i^T u_j + \sum_i \sum_j a_j b_i v_i^T u_j.
\end{aligned}$$

Therefore, the right-hand side of Equation (8) also equals (9). □

Corollary 6.10 *The dimension of the space of infinitesimal motions of* $\mathbf{R}^d$ *is* $\binom{d+1}{2}$.

Corollary 6.11 *A d-dimensional framework G in $\mathbf{R}^d$ is infinitesimally rigid if and only if its motion space has dimension $\binom{d+1}{2}$. In this case, every infinitesimal motion is determined by its restriction to any d affinely independent vertices of G.*

Corollary 6.12 *Let G and G' be two infinitesimally rigid d-dimensional frameworks in $\mathbf{R}^d$ that have d affinely independent vertices in common. Then the union of G and G' is also infinitesimally rigid.*

Given a framework G in $\mathbf{R}^d$ with vertices $v_1, \ldots, v_n$, we can write the conditions for $u_1, \ldots, u_n$ to be an infinitesimal motion in matrix form. Let $f_0 = n$ and let A be a matrix with f_1 rows, one for each edge of G, and df_0 columns, d for each vertex of G. In the row corresponding to edge ij, place the row vector $(v_j - v_i)^T$ in the d columns corresponding to v_i, and $v_i - v_j$ in the d columns corresponding to v_j. The remaining entries of A are set to 0. Let $u^T = (u_1^T, \ldots, u_n^T)$. Then $u_1, \ldots, u_n$ is an infinitesimal motion if and only if $Au = O$ (you should check this).

Theorem 6.13 *The motion space of a framework is the nullspace of its matrix A.*

6.3 Stress

The motion space provides a geometrical interpretation of the nullspace of A. What about the left nullspace? An element λ of the left nullspace assigns a number λ_{ij} $(= \lambda_{ji})$ to each edge of the framework, and the statement $\lambda^T A = O^T$ is equivalent to the equations

$$\sum_{j:ij \in E} \lambda_{ij}(v_j - v_i) = O \text{ for every vertex } i. \tag{10}$$

This can be regarded as a set of equilibrium conditions (one at each vertex) for the λ_{ij}, which may be thought of as forces or *stresses* on the edges of the framework. The left nullspace of A is called the *stress space*, and A itself is sometimes called the *stress matrix*.

Exercise 6.14 Give some examples of stresses for two-dimensional frameworks.

Putting together everything we know, we have several ways to test infinitesimal rigidity:

Theorem 6.15 *The following are equivalent for a d-dimensional framework G with f_0 vertices and f_1 edges:*

1. *The framework G is infinitesimally rigid.*

2. *The dimension of the motion space of G (the nullspace of A) is* $\binom{d+1}{2}$.

3. *The rank of A is* $df_0 - \binom{d+1}{2}$.

4. *The dimension of the stress space of G (the left nullspace of A) is* $f_1 - df_0 + \binom{d+1}{2}$.

Corollary 6.16 *Let P be a simplicial d-polytope. Then P is infinitesimally rigid if and only if the dimension of its stress space equals $g_2(P) := h_2(P) - h_1(P)$.*

Proof. From Equation (4), $h_1(P) = -df_{-1} + f_0$ and $h_2(P) = \binom{d}{d-2} f_{-1} - (d-1)f_0 + f_1$, so $h_2(P) - h_1(P) = \binom{d}{2} - (d-1)f_0 + f_1 + d - f_0 = f_1 - df_0 + \binom{d+1}{2}$. $\square$

6.4 Infinitesimal Rigidity of Simplicial Polytopes

A good reference for this section is the paper by Roth [Rot81]. Cauchy [Cau13] proved that simplicial 3-polytopes are rigid. Dehn [Deh16] used the stress matrix to prove the stronger result that these polytopes are infinitesimally rigid.

Theorem 6.17 (Dehn, 1916) *Let P be a simplicial convex 3-polytope. Then P admits only the trivial stress $\lambda_{ij} = 0$ for all edges ij.*

PROOF. This proof is my slight modification of the proof presented in Roth [Rot81].

Suppose there is a non-trivial stress. Label each edge $ij \in E$ with the sign $(+, -, 0)$ of λ_{ij}. Suppose there is a vertex v such that all edges incident to it are labeled 0. Then delete v and take the convex hull of the remaining vertices. The result cannot be two-dimensional, because it is clear that there can be no non-trivial stress on the edges of a single polygon (do you see why?). So the result is three-dimensional. If it is not simplicial, triangulate the non-simplicial faces arbitrarily, labeling the new edges 0. Repeat this procedure until you have a triangulated 3-polytope Q (possibly with some coplanar triangles) such that every vertex is incident to at least one nonzero edge. Note that every nonzero edge of Q is an edge of the original polytope P.

Now in each corner of each triangle of Q place the label 0 if the two edges meeting there are of the same sign, 1 if they are of opposite sign, and 1/2 if one is zero and the other nonzero.

Claim 1. The sum of the corner labels at each vertex v is at least four. First, because v is a vertex of P, the nonzero edges of P incident to v cannot all have the same sign. Consider now the sequence of changes in signs of just the nonzero edges of P incident to v as we circle around v. If there were only two changes in sign, the positive edges could be separated from the negative edges by a plane passing through v, since no three edges incident to v in P are coplanar. So there must be at least four changes in sign. The claim for the corner labels in Q now follows easily.

Claim 2. The sum of the three corner labels for each triangle of Q is at most two. Just check all the possibilities of the edge and corner labels for a single triangle.

Now consider the sum S of all the corner labels of Q. By Claim 1 the sum is at least $4f_0$, where f_0 is the number of vertices of Q. By Claim 2 the sum is at most $2f_2$, where f_2 is the number of triangles of Q. But $f_0 - f_1 + f_2 = 2$ by Euler's Relation, where f_1 is the number of edges of Q. Also, each triangle has three edges and each edge is in two triangles, so $3f_2 = 2f_1$. Therefore $f_2 = 2f_0 - 4$. So $4f_0 \leq S \leq 4f_0 - 8$ yields a contradiction. $\square$

Corollary 6.18 *Simplicial convex 3-polytopes are infinitesimally rigid.*

PROOF. The dimension of the stress space equals 0, which equals $h_2(P) - h_1(P)$ by the Dehn-Sommerville Equations. The result follows by Corollary 6.16. $\square$

Corollary 6.18 tells us that if we build the geometric skeleton of a simplicial 3-polytope out of bars which meet at flexible joints then the structure will be infinitesimally rigid. Similarly the structure will be infinitesimally rigid if we build the boundary of the polytope out of triangles which meet along flexible edges. However, if the structure is not convex, it might flex infinitesimally, and there are easy examples of this. Connelly [Con77] showed the truly remarkable fact that that there are simplicial 2-spheres immersed in $\mathbf{R}^3$ that have a finite real flex—a motion that is not just infinitesimal. Sabitov [Sab95] proved that during such a flex the enclosed volume remains constant (the "Bellows" Theorem), a result that was extended to all triangulated orientable flexible surfaces by Connelly, Sabitov, and Walz [CSW97]. Of course, if a convex 3-polytope is not simplicial, then its skeleton may flex (consider the cube).

Whiteley [Whi84] extended the rigidity theorem to higher dimensions.

Theorem 6.19 (Whiteley, 1984) *Simplicial convex d-polytopes, $d \geq 3$, are infinitesimally rigid.*

PROOF. We proceed by induction on d. The result is true for $d = 3$, so assume P is a simplicial d-polytope, $d > 3$. Let v_0 be any vertex of P. Define G to be the framework consisting of all vertices and edges contained in all facets containing v_0 (the vertices and edges of the *closed star* of v_0). Let the edges of G be indexed by E. Construct Q, a vertex figure of P at v_0.

Claim 1. The stress spaces of G (regarded as a d-dimensional framework) and of Q (regarded as a $(d-1)$-dimensional framework) have the same dimension. Stresses are unaffected by Euclidean motions and by scaling of the framework, so assume without loss of generality that $v_0 = O$ and the neighbors of v_0 in G have coordinates $(v_1, a_1), \ldots, (v_m, a_m)$, with $a_1, \ldots, a_m > 1$. Assume that the hyperplane used to construct the vertex figure has equation $x_d = 1$. Hence the vertices of Q are v_i/a_i, $i = 1, \ldots, m$, regarded as a $(d-1)$-polytope in $\mathbf{R}^{d-1}$. Let λ be a stress on G. For $i = 1, \ldots, m$, the equilibrium conditions (10) imply

$$\left[\sum_{j \neq 0 : ij \in E} \lambda_{ij}(v_i - v_j)\right] + \lambda_{i0} v_i = O,$$

and

$$\left[\sum_{j \neq 0 : ij \in E} \lambda_{ij}(a_i - a_j)\right] + \lambda_{i0} a_i = 0.$$

Hence

$$\lambda_{i0} = -\frac{1}{a_i} \sum_{j\neq 0: ij \in E} \lambda_{ij}(a_i - a_j). \tag{11}$$

Define $\overline{\lambda}_{ij} = a_i a_j \lambda_{ij}$ for every edge $v_i v_j$ of Q. We can verify that $\overline{\lambda}$ is a stress on Q. For $i = 1, \ldots, m$,

$$\begin{aligned}
\sum_{j\neq 0: ij\in E} \overline{\lambda}_{ij}\left(\frac{v_i}{a_i} - \frac{v_j}{a_j}\right) &= \sum_{j\neq 0: ij\in E} \lambda_{ij}(a_j v_i - a_i v_j) \\
&= \sum_{j\neq 0: ij\in E} \lambda_{ij}(a_j v_i - a_i v_i) + \sum_{j\neq 0: ij\in E} \lambda_{ij}(a_i v_i - a_i v_j) \\
&= \left[\sum_{j\neq 0: ij\in E} \lambda_{ij}(a_j - a_i)\right] v_i + a_i \sum_{j\neq 0: ij\in E} \lambda_{ij}(v_i - v_j) \\
&= a_i \lambda_{i0} v_i + a_i \sum_{j\neq 0: ij\in E} \lambda_{ij}(v_i - v_j) \\
&= a_i(O) \\
&= O.
\end{aligned}$$

Conversely, starting with a stress $\overline{\lambda}$ on Q, we can reverse this process, defining $\lambda_{ij} = \overline{\lambda}_{ij}/(a_i a_j)$ for $ij \in E$, $i, j \neq 0$, and using Equation (11) to define λ_{i0} for all i. In this manner we obtain a stress for G.

Claim 2. The framework G is infinitesimally rigid. The simplicial d-polytope Q is infinitesimally rigid by induction, so by Theorem 6.15 the dimension of the stress space of Q is $f_1(Q) - (d-1)f_0(Q) + \binom{d}{2}$. Claim 1 implies that this is also the dimension of the stress of G. Hence

$$\begin{aligned}
f_1(G) - df_0(G) + \tbinom{d+1}{2} &= (f_1(Q) + f_0(Q)) - d(f_0(Q) + 1) + \tbinom{d+1}{2} \\
&= f_1(Q) - (d-1)f_0(Q) + \tbinom{d}{2} \\
&= \text{the dimension of the stress space of } Q \\
&= \text{the dimension of the stress space of } G.
\end{aligned}$$

Therefore G is infinitesimally rigid by Theorem 6.15.

Now consider any two adjacent vertices v and v' of G. The frameworks of their closed stars are each infinitesimally rigid by Claim 2, and share d affinely independent vertices (those on any common facet). Thus the union of these two frameworks is infinitesimally rigid by Corollary 6.12. Therefore repeated application of Theorem 6.12 implies that P is infinitesimally rigid. □

6.5 Kalai's Proof of the Lower Bound Theorem

Theorem 6.20 (Kalai, 1987) *Let P be a simplicial d-polytope with n vertices. Then $f_j(P) \geq f_j(P(n,d))$, $j = 0, \ldots, d-1$.*

Proof. Since P is infinitesimally rigid, the Theorem dimension of its stress space equals $g_2 := h_2(P) - h_1(P)$. Hence this quantity is nonnegative, and so $h_2(P) \geq h_1(P) = n - d$. Therefore $f_1(P) \geq f_1(P(n,d))$.

To establish the result for higher-dimensional faces, Kalai uses the "McMullen-Perles-Walkup" (MPW) reduction. I am going to quote this proof almost verbatim from Kalai's paper, so will use his notation. Let $\phi_k(n,d) := f_k(P(n,d)$. For a simplicial d-polytope C with n vertices define $\gamma(C) = f_1(C) - \phi_1(n,d) = g_1(C)$. Thus, for $d \geq 3$, $\gamma(C) = f_1(C) - dn + \binom{d+1}{2}$ and for $d = 2$, $\gamma(C) = f_1(C) - n$. Define also

$$\gamma_k(C) = f_k(C) - \phi_k(n,d).$$

Let S be any face of $\mathrm{bd}\, C$ with k vertices. The *link* of S in C is defined to be

$$\mathrm{lk}(S,C) := \{T : T \text{ is a face of } \mathrm{bd}\, C,\ T \cap S = \emptyset,\ \mathrm{conv}\,(T \cup S) \text{ is a face of } \mathrm{bd}\, C\}.$$

It is known that $\mathrm{lk}(S,C)$ is isomorphic to set of boundary faces of some $(d-k)$-polytope (take repeated vertex figures). Define

$$\gamma^k(C) = \sum \{\gamma(\mathrm{lk}(S,C)) : S \in C,\ |S| = k\}.$$

Thus, $\gamma_1(C) = \gamma^0(C) = \gamma(C)$.

Proposition 6.21 *Let C be a simplicial d-polytope, and let k, d be integers, $1 \leq k \leq d-1$. There are positive constants $w_i(k,d)$, $0 \leq i \leq k-1$, such that*

$$\gamma_k(C) = \sum_{i=0}^{k-1} w_i(k,d) \gamma^i(C). \tag{12}$$

Proof. First note that

$$(k+1) f_k(C) = \sum_{i=1}^{n} f_{k-1}(\mathrm{lk}(v,C)). \tag{13}$$

Put $\phi_k(n,d) = a_k(d)n + b_k(d)$. (Thus, $a_k(d) = \binom{d}{k}$ for $1 \le k \le d-2$ and $a_{d-1}(d) = d-1$.) Easy calculation gives

$$2\left(dn - \binom{d+1}{2}\right) a_{k-1}(d-1) + nb_{k-1}(d-1) = (k+1)\phi_k(n,d).$$

Let C be a simplicial d-polytope, $d \ge 3$, with n vertices $v_1, \ldots, v_n$. Assume that the degree of v_i in C is n_i (i.e., $f_0(\mathrm{lk}(v_i,c)) = n_i$). Note that $\sum_{i=1}^n n_i = 2f_1(C) = 2(dn - \binom{d+1}{2} + \gamma(C))$. Therefore

$$\begin{aligned}\sum_{i=1}^n \phi_{k-1}(n_i, d-1) &= a_{k-1}(d-1)\sum_{i=1}^n n_i + nb_{k-1}(d-1) \\ &= a_{k-1}(d-1)2\left(dn - \binom{d+1}{2}\right) + 2a_{k-1}(d-1)\gamma(C) + nb_{k-1}(d-1) \\ &= (k+1)\phi_k(n,d) + 2a_{k-1}(d-1)\gamma(C).\end{aligned} \tag{14}$$

From (13) and (14) we get

$$(1+k)\gamma_k(C) = 2a_{k-1}(d-1)\gamma(C) + \sum_{i=1}^n \gamma_{k-1}(\mathrm{lk}(v_i,C)). \tag{15}$$

Repeated applications of formula (15) give (12). The value of $w_i(k,d)$ is

$$w_i(k,d) = \begin{cases} 2(a_{k-i-1}(d-i-1))/(k+1)\binom{k}{i}, & 0 \le i \le k-2, \\ 2/(k+1)k, & i = k-1. \end{cases}$$

Corollary 6.22 (The MPW-reduction) *Let $d \ge 2$ be an integer. Let C be a simplicial d-polytope with n vertices, such that $\gamma(\mathrm{lk}(S,C)) \ge 0$ for every face S of $\mathrm{bd}\, C$, $|S| < k$. Then*

1. *$f_k(C) \ge \phi_k(n,d)$.*
2. *If $f_k(C) = \phi_k(n,d)$ then $\gamma(C) = 0$.*

□

Exercise 6.23 Check the details of the above Proposition and Corollary. □

Kalai [Kal87] discusses the extension of the Lower Bound Theorem to more general classes of objects.

The important insight of Kalai's proof is that $h_2 - h_1$ is nonnegative for a simplicial d-polytope, $d \ge 3$, because it *counts* something; namely, the dimension of a certain vector space. It turns out that $h_i - h_{i-1}$ is also nonnegative, $i = 3, \ldots, \lfloor d/2 \rfloor$ as well, and it is possible to generalize appropriately the notion of stress to capture this result.

6.6 The Stress Polynomial

Suppose we have a stress λ on a d-dimensional framework in $\mathbf{R}^d$ with vertices indexed by $1,\ldots,n$ and edges indexed by E. Define $\lambda_{ij}=0$ if $i\neq j$, $ij\notin E$. Taking $\lambda_{ij}=\lambda_{ji}$ if $i\neq j$, we define

$$\lambda_{jj} := -\sum_{i:i\neq j}\lambda_{ij},\ j=1,\ldots,n.$$

Then define

$$b(x_1,\ldots,x_n) := \sum_{i,j:i\neq j}\lambda_{ij}x_ix_j+\sum_j \frac{\lambda_{jj}}{2}x_j^2.$$

This *stress polynomial* (see Lee [Lee96]) $b(x)$ captures the definition of stress in the following way. Let $\overline{M}$ be the $(d+1)\times n$ matrix

$$\overline{M} := \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{bmatrix}.$$

Theorem 6.24 *λ is a stress if and only if $\overline{M}\nabla b = O$.*

In this theorem, $\nabla b(x) := (\frac{\partial}{\partial x_1}b(x),\ldots,\frac{\partial}{\partial x_n}b(x))$, and we are regarding $\overline{M}\nabla b$ as a member of $(\mathbf{R}[x_1,\ldots,x_n])^{d+1}$.

Proof. Starting with the equilibrium conditions for stress,

$$\sum_{i:i\neq j}\lambda_{ij}(v_i-v_j)=O,\ j=1,\ldots,n,$$

$$\sum_{i:i\neq j}\lambda_{ij}v_i+\left(-\sum_{i:i\neq j}\lambda_{ij}\right)v_j=O,\ j=1,\ldots,n,$$

$$\sum_{i:i\neq j}\lambda_{ij}v_i+\lambda_{jj}v_j=O,\ j=1,\ldots,n.$$

$$\sum_{i=1}^n\lambda_{ij}v_i=O,\ j=1,\ldots,n.$$

Also, obviously,

$$\sum_{i:i\neq j}\lambda_{ij}+\lambda_{jj}=0,\ j=1,\ldots,n,$$

$$\sum_{i=1}^n\lambda_{ij}=0,\ j=1,\ldots,n.$$

Also,

$$\frac{\partial}{\partial x_i} b(x) = \sum_{j=1}^{n} \lambda_{ij} x_j, \ i = 1, \ldots, n.$$

Therefore

$$\begin{aligned}
\overline{M} \nabla b &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_{ij} x_j \right) \begin{bmatrix} v_i \\ 1 \end{bmatrix} \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \lambda_{ij} \begin{bmatrix} v_i \\ 1 \end{bmatrix} \right) x_j \\
&= \begin{bmatrix} O \\ 0 \end{bmatrix} x_j \\
&= \begin{bmatrix} O \\ 0 \end{bmatrix}.
\end{aligned}$$

□

Some remarks:

1. Every nonzero coefficient of the stress polynomial is associated naturally to a certain face of the framework.

2. The coefficients of the square-free terms uniquely determine the coefficients of the x_i^2 terms.

3. If we define the matrix

$$M := \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

then the condition of Theorem 6.24 can be written as $M \nabla b(x) = O$ and $\omega b(x) = 0$, where

$$\omega := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}.$$

7 Simplicial Complexes

Since boundaries of simplicial polytopes provide examples of simplicial complexes, we now study what we can determine about f-vectors and h-vectors of various classes of simplicial complexes. Stanley's book [Sta96] is a good source of material for this section, and provides further references.

Let V be a finite set. An *(abstract) simplicial complex* Δ is a nonempty collection of subsets of V such that $F \subset G \in \Delta$ implies $F \in \Delta$. In particular, $\emptyset \in \Delta$. For $F \in \Delta$ we say F is a *face* of Δ and the *dimension* of F, $\dim F$, equals $\operatorname{card}(F) - 1$. We define $\dim \Delta := \max\{\dim F : F \in \Delta\}$ and refer to a simplicial complex of dimension $d-1$ as a *simplicial* $(d-1)$*-complex*. Faces of dimension 0, 1, $d-2$, and $d-1$ are called *vertices edges*, *subfacets* or *ridges*, and *facets* of Δ, respectively. For simplicial $(d-1)$-complex Δ we define $f_j(\Delta)$ to be the number of j-dimensional (j*-faces*) of Δ, and its f*-vector* to be $f(\Delta) := (f_0(\Delta), f_1(\Delta), \ldots, f_{d-1}(\Delta))$, and then use the same equation (4) for simplicial d-polytopes to define the h*-vector* of Δ.

Exercise 7.1 Suppose Δ is a simplicial complex on $V = \{1, \ldots, n\}$. Prove that there exists a positive integer e and points $v_1, \ldots, v_n \in \mathbf{R}^e$ such that $\operatorname{conv}\{v_i : i \in F\} \cap \operatorname{conv}\{v_i : i \in G\} = \operatorname{conv}\{v_i : i \in F \cap G\}$. In this way we can realize any simplicial complex geometrically. □

7.1 The Kruskal-Katona Theorem

For positive integers a and i, a can be expressed uniquely in the form

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. This is called the *$i$-canonical representation of a.*

Exercise 7.2 Prove that the i-canonical representation exists and is unique. □

From this representation define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}$$

where $\binom{k}{\ell} = 0$ if $\ell > k$. Define also $0^{(0)} = 0$.

Kruskal [Kru63] characterized the f-vectors of simplicial complexes in 1963. Katona [Kat68] found a shorter proof in 1968. The theorem is also a consequence of the generalization by Clements and Lindström [CL69].

Theorem 7.3 (Kruskal-Katona) *The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ of positive integers is the f-vector of some simplicial $(d-1)$-dimensional complex Δ if and only if*

1. $f_{-1} = 1$, *and*
2. $f_j \leq f_{j-1}^{(j)}$, $j = 1, 2, \ldots, d-1$.

Proof. (Sketch.)

Sufficiency: Let $V = \{1, 2, \ldots\}$. Let $V^i = \{F \subseteq V : |F| = i\}$. Order the sets in V^i *reverse lexicographically*. That is, for $F, G \in V^i$, $F \neq G$, define $F < G$ if there exists a k such that $k \notin F$, $k \in G$, and $i \in F$ if and only if $i \in G$ for all $i > k$. For all j choose the first f_{j-1} sets of V^j. The conditions will force the resulting collection to be a simplicial complex.

Example:

1	6	13	10
$\underline{\emptyset}$	1	12	123
	2	13	124
	3	23	134
	4	14	234
	5	24	125
	$\underline{6}$	34	135
		15	235
		25	145
		35	245
		45	$\underline{345}$
		16	126
		26	136
		$\underline{36}$	236
		46	146
		56	246
			346
			156
			256
			356
			456

Necessity: Given simplicial complex Δ. By application of a certain "shifting" or "compression" operation, transform it to a reverse lexicographic simplicial complex with the same f-vector. Then verify that the conditions must hold. $\square$

Corollary 7.4 *f-vectors of simplicial d-polytopes must satisfy the Kruskal-Katonal conditions.*

7.2 Order Ideals of Monomials

We will soon see that understanding how to count monomials will help in investigating h-vectors of certain simplicial complexes. Let X be the finite set $\{x_1, \ldots, x_n\}$. An *order ideal of monomials* is a nonempty set M of monomials $x_1^{b_1} \cdots x_n^{b_n}$ in the variables x_i such that $m|m' \in M$ implies $m \in M$. In particular, $1 = x_1^0 \cdots x_n^0 \in M$. Let $h_i(M)$ be the number of monomials in M of degree i. The sequence $h = (h_0(M), h_1(M), \ldots)$ is called an *M-sequence*, or an *M-vector* if it terminates $(h_0, \ldots, h_d)$ for some d.

For positive a and i, use the i-canonical representation of a to define

$$a^{\langle i \rangle} = \binom{a_i + 1}{i + 1} + \binom{a_{i-1} + 1}{i} + \cdots + \binom{a_j + 1}{j + 1}.$$

Define also $0^{\langle i \rangle} = 0$.

Stanley [Sta78] (see also [Sta75b, Sta77, Sta96]) proved the following characterization of M-sequences of order ideals of monomials, which is analogous to the Kruskal-Katona Theorem, as one piece of a much larger program in which he established, elucidated and exploited new connections between combinatorics and commutative algebra.

Theorem 7.5 (Stanley) $(h_0, h_1, h_2, \ldots)$, *a sequence of nonnegative integers, is an M-sequence if and only if*

1. $h_0 = 1$, *and*
2. $h_{i+1} \leq h_i^{\langle i \rangle}$, $i = 1, 2, 3, \ldots$.

Proof. (Sketch.)

Sufficiency: Let M^i be the set of all monomials of degree i. Order the monomials in M^i *reverse lexicographically*. That is, for $m, m' \in M^i$, $m \neq m'$, $m = x_1^{b_1} \cdots x_n^{b_n}$, $m' = x_1^{b'_1} \cdots x_n^{b'_n}$, we say $m < m'$ if there is some k such that $b_k < b'_k$ and $b_i = b'_i$ for all $i > k$. For all i choose the first h_i monomials of M^i. The conditions will force the resulting collection to be an order ideal of monomials.

Example:

$$
\begin{array}{llll}
1 & 3 & 4 & 2 \\
\hline
\underline{1} & x_1 & x_1^2 & x_1^3 \\
 & x_2 & x_1x_2 & \underline{x_1^2x_2} \\
 & \underline{x_3} & x_2^2 & x_1x_2^2 \\
 & & \underline{x_1x_3} & x_2^3 \\
 & & x_2x_3 & x_1^2x_3 \\
 & & x_3^2 & x_1x_2x_3 \\
 & & & x_2^2x_3 \\
 & & & x_1x_3^2 \\
 & & & x_2x_3^2 \\
 & & & x_3^3
\end{array}
$$

Complete details can be found in Billera and Lee [BL81].

Necessity: Given an order ideal of monomials, "shift" or "compress" it to a reverse lexicographic order ideal with the same M-sequence. Then verify that the conditions must hold. The fact that the compression technique results in an order ideal of monomials is due to Macaulay [Mac27] (hence Stanley's choice of "M" in "M-sequence"). Clements and Lindström [CL69] provide a more accessible proof of a generalization of Macaulay's theorem. □

7.3 Shellable Simplicial Complexes

Let Δ be a simplicial $(d-1)$-complex. We say that Δ is *shellable* if it is pure (every face of Δ is contained in a facet), with the property that the facets can be ordered $F_1, \ldots, F_m$ such that for $k = 2, \ldots, m$ there is a unique minimal nonempty face G_k of $\mathcal{F}(F_k)$ that is not in $S_{k-1} := \mathcal{F}(F_1) \cup \cdots \cup \mathcal{F}(F_{k-1})$. See Exercise 4.9, in which we conclude that for every k,

$$
h_i(S_k) = \begin{cases} h_i(S_{k-1}) + 1, & i = f_0(G_k), \\ h_i(S_{k-1}), & \text{otherwise.} \end{cases}
$$

Stanley [Sta77, Sta78] stated the following theorem:

Theorem 7.6 (Stanley) *$(h_0, \ldots, h_d) \in \mathbf{Z}_+^{d+1}$ is the h-vector of some shellable simplicial $(d-1)$-complex if and only if it is an M-vector.*

Proof. We will sketch Stanley's construction for sufficiency, leaving the necessity of the conditions for later. I think the construction first appeared in Lee [Lee81]. This type of construction was the core of the combinatorial portion of the proof in [BL81]. See also [Lee84] for a slight generalization.

Let $V = \{1, 2, 3, \ldots\}$. Let V^i be the collection of all subsets F of V of cardinality d such that $1, \ldots, d - i \in F$ but $d - i + 1 \notin F$. For all i choose the first h_i sets in V^i, using reverse lexicographic order. We claim that these are the facets of Δ, and they are shellable in reverse lexicographic order. Further, if a chosen F is in V^i then it contributes to $h_i(\Delta)$ during the shelling.

Associate with each facet $F = \{i_1, \ldots, i_d\}$ ($i_1 < \cdots < i_d$) the monomial $m(F)$ $x_{i_1-1}x_{i_2-2}\cdots x_{i_d-d}$, where we interpret $x_0 = 1$. Then $F \in V^i$ if and only if $\deg m(F) = i$. By the proof of Theorem 7.5, the selected monomials will form an order ideal that is also closed (within each degree) under reverse lexicographic order. We call such a collection of monomials a *lexicographic order ideal*.

Example: $d = 3$, $h = (1, 3, 4, 2)$. Chosen facets are marked with an asterisk.

$m(F)$	i	1	2	3	4	5	6
1	0^*	1	2	3			
x_1	1^*	1	2		4		
x_1^2	2^*	1		3	4		
x_1^3	3^*		2	3	4		
x_2	1^*	1	2			5	
x_1x_2	2^*	1		3		5	
$x_1^2x_2$	3^*		2	3		5	
x_2^2	2^*	1			4	5	
$x_1x_2^2$	3		2		4	5	
x_2^3	3			3	4	5	
x_3	1^*	1	2				6
x_1x_3	2^*	1		3			6
$x_1^2x_3$	3		2	3			6
x_2x_3	2	1			4		6
$x_1x_2x_3$	3		2		4		6
$x_2^2x_3$	3			3	4		6
x_3^2	2	1				5	6
$x_1x_3^2$	3		2			5	6
$x_2x_3^2$	3			3		5	6
x_3^3	3				4	5	5

Let $F = \{i_1, \ldots, i_d\}$ be a facet of V^i in Δ. Choose $G = F \setminus \{1, \ldots, d - i\}$. It is not hard to prove that no facet preceding F in reverse lexicographic order contains G.

Choose $j > d - i$. Find $k_j := \max\{k \notin F : k < i_j\}$ and define $F_j := (F \setminus \{i_j\}) \cup \{k_j\}$. Obviously F and F_j are neighbors, i.e., share $d - 1$ elements. Knowing that the monomials associated with the facets in Δ form a lexicographic order ideal, we can also verify that $F_j \in \Delta$ for every j.

From the above analysis it is possible to conclude that G is the unique minimal new face of F added to Δ when F is shelled onto the preceding facets of Δ. $\square$

8 The Stanley-Reisner Ring

8.1 Overview

To finish the proof of the previous section and show that the h-vector of a shellable simplicial complex Δ is an M-vector, we need to show how to construct a suitable order ideal of monomials from Δ. This is facilitated by certain algebraic tools developed by Stanley. A good general reference is [Sta96].

Let Δ be a simplicial $(d-1)$-complex with n vertices $1, \ldots, n$. Consider the polynomial ring $R = \mathbf{R}[x_1, \ldots, x_n]$. There is a natural grading of $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ by degree, where R_i consists of only those polynomials, each of whose terms have degree exactly i. For a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ in R we define the *support* of m to be $\text{supp}(m) = \{i : a_i > 0\}$. Let I be the ideal of R generated by monomials m such that $\text{supp}(m) \notin \Delta$. The *Stanley-Reisner ring* or *face ring* of Δ is $A_\Delta := R/I$. Informally, we do calculations as in R but set any monomial to zero whose support does not correspond to a face.

The ring A_Δ is also graded $A_\Delta = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ by degree. We will see that $\sum \dim A_i t^i = f(\Delta, \frac{t}{1-t})$. Stanley proved that if Δ is shellable, then there exist d elements $\theta_1, \ldots, \theta_d \in A_1$ (a *homogeneous system of parameters*) such that θ_i is not a zero-divisor in $A_\Delta/(\theta_1, \ldots, \theta_{i-1})$, $i = 1, \ldots, d$. Let $B = A_\Delta/(\theta_1, \ldots, \theta_d) = B_0 \oplus B_1 \oplus \cdots \oplus B_d$. Then $\sum \dim B_i t^i = (1-t)^d f(\Delta, \frac{t}{1-t}) = h(\Delta, t)$. So $\dim B_i = h_i$, $i = 0, \ldots, d$. Macaulay [Mac27] proved that there exists a basis for B as an $\mathbf{R}$-vector space that is an order ideal of monomials. Theorem 7.6 then follows immediately from Theorem 7.5.

The existence of the θ_i means that A_Δ is *Cohen-Macaulay* and as a consequence $h(\Delta)$ is an M-vector. Reisner [Rei76] characterized the class of Cohen-Macaulay complexes, those simplicial complexes Δ for which A_Δ is a Cohen-Macaulay ring:

Theorem 8.1 (Reisner) *A simplicial complex Δ is Cohen-Macaulay if and only if for all $F \in \Delta$, $\dim \tilde{H}_i(\text{lk}_\Delta F, \mathbf{R} = 0$ when $i < \dim \text{lk}_\Delta F$.*

In particular, simplicial complexes that are topological balls (*simplicial balls*) and spheres (*simplicial spheres*), whether shellable or not, are Cohen-Macaulay.

8.2 Shellable Simplicial Complexes are Cohen-Macaulay

Let Δ be a simplicial $(d-1)$-complex with n vertices $1, \ldots, n$, and consider the ring $A_\Delta = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$.

Theorem 8.2 (Stanley) *The dimension of A_ℓ as a vector space over $\mathbf{R}$ is $H_\ell(\Delta)$, where*

$$H_\ell(\Delta) = \begin{cases} 1, & \ell = 0, \\ \\ \displaystyle\sum_{j=0}^{\ell-1} f_j(\Delta)\binom{\ell-1}{j}, & \ell > 0, \end{cases}$$

(taking $f_j(\Delta) = 0$ if $j \geq d$).

Proof. We need to show that the number of nonzero monomials of degree ℓ in A_Δ is $H_\ell(\Delta)$. Let F be a face of dimension j (hence cardinality $j+1$). By Exercise 3.11 the number of monomials m of degree ℓ such that $\operatorname{supp}(m) = \operatorname{supp}(F)$ is $\binom{\ell-1}{j}$. The result now follows easily. □

Let T be any $d \times n$ matrix such that every $d \times d$ submatrix associated with a facet of Δ is invertible. If Δ happens to be the boundary complex of a simplicial d-polytope $P \subset \mathbf{R}^d$ containing the origin in its interior, then we can take T to be the matrix whose columns are the coordinates of the vertices of P. For $i = 1, \ldots, d$ define $\theta_i = t_{i1}x_1 + \cdots + t_{in}x_n \in A_1$. That is to say, θ_i is a linear expression whose coefficients can be read off from row i of T.

Theorem 8.3 (Reisner-Stanley) *There exist monomials $\eta_1, \ldots, \eta_m \in A$ such that every member y of A has a unique representation of the form $y = \sum_{i=1}^{m} p_i\eta_i$, where the p_i are polynomials in the θ_i.*

Proof. We sketch the proof of Kind and Kleinschmidt [KK76]. Let $F_1, \ldots, F_m$ be a shelling of the facets of Δ and define S_j to be the collection of all faces of Δ in $F_1, \ldots, F_j$, $j = 1, \ldots, m$. Define the Stanley-Reisner ring A_{S_j} of S_j in the natural way. We will prove by induction on j that A_{S_j} has the property of the theorem.

For facet F_j, the columns in T corresponding to the vertices of F_j determine a $d \times d$ submatrix U of T. Multiply T on the left by U^{-1} to get the matrix T'. For $i = 1, \ldots, d$ define $\theta_i' := t_{i1}'x_1 + \cdots + t_{in}'x_n \in A_1$. Then the θ_i' are linear combinations of the θ_i and vice versa since the relations are invertible.

First suppose $j = 1$. For convenience, suppose F_1 contains the vertices $1, \ldots, d$. Then $x_i = 0$, $i = d+1, \ldots, n$ and $\theta_i' = x_i$, $i = 1, \ldots, d$. The elements of A_{S_1} are precisely the polynomials in the variables $x_1, \ldots, x_d$. Since $x_1^{a_1} \cdots x_d^{a_d} = {\theta'}_1^{a_1} \cdots {\theta'}_d^{a_d}$, choosing $\eta_1 = 1$ we can see that every member y of A_{S_1} has a representation of the form $y = p_1'\eta_1$ where p_1' is a polynomial in the θ_i'. Transforming back to the θ_i, $p_1'(\theta_1', \ldots, \theta_d') = p_1(\theta_1, \ldots, \theta_d)$, a polynomial in the θ_i. To show that the representation is unique, suppose $p_1(\theta_1, \ldots, \theta_d) = 0$ for some polynomial p_1 in the θ_i. Transforming the θ_i to θ'_i, we have a polynomial p'_1 in the $\theta'_i = x_i$ which equals 0. Therefore p'_1, and hence p_1, must be the zero polynomial.

Now suppose $j > 1$. Let G_j be the unique minimal face of F_j that is not present in S_{j-1}, and let $k := \operatorname{card} G_j$. For convenience, assume F_j contains the vertices $1, \ldots, d$ and G_j contains the vertices $1, \ldots, k$. Let $\eta_j := x_1 \cdots x_k$.

Consider any nonzero monomial m in A_{S_j} that is divisible by η_j. Then the support of m contains G_j and can therefore consist only of variables from among $x_1, \ldots, x_d$ since all faces in S_j containing G_j are subsets of F_j. Then $m = m'\eta_j$ where $m' = x_1^{a_1} \cdots x_d^{a_d}$. It is now easy to check that $m'\eta_j = \theta'^{a_1}_1 \cdots \theta'^{a_d}_d \eta_j$ since upon expanding, all monomials are divisible by η_j and those containing variables other than $x_1, \ldots, x_d$ are zero in A_{S_j}. From this, transforming the θ'_i to the θ_i, we can see that m can be expressed in the form $p_j\eta_j$, where p_j is a polynomial in the θ_i. Since we can handle monomials divisible by η_j, it is now easy to see that any $y \in A_{S_j}$ that is divisible by η_j can be expressed as a product of a polynomial in the θ_i and the monomial η_j.

Now consider any $y \in A_{S_j}$ such that no monomial in y is divisible by η_j. Then, regarding y as a member of $A_{S_{j-1}}$, $y = \sum_{i=1}^{j-1} p_i\eta_i$. But this may no longer be true in A_{S_j} since after expanding the sum there may be some monomials left over that are divisible by η_j, which were zero in $A_{S_{j-1}}$, but not in A_{S_j}. So $y = \sum_{i=1}^{j-1} p_i\eta_i + w$, where w is divisible by η_j. Now find a representation for w as in the preceding paragraph.

It remains to show that the representations are unique. Assume that $\sum_{i=1}^{j} p_i\eta_i = 0$. Setting all terms divisible by η_j equal to zero, it must be the case that $\sum_{i=1}^{j-1} p_i\eta_i = 0$ in $A_{S_{j-1}}$. So each of the polynomials $p_1, \ldots, p_{j-1}$ is the zero polynomial by induction. Hence $p_j\eta_j = 0$ in A_{S_j}. Transforming the θ_i to the θ'_i, we have $p'_j\eta_j = 0$ for some polynomial p'_j in the θ'_i. But for each term in the expansion, $\theta'^{a_1}_1 \cdots \theta'^{a_d}_d \eta_j = x_1^{a_1} \cdots x_d^{a_d} \eta_j$, from which one readily sees that p'_j must be the zero polynomial. Transforming the θ'_i back to the θ_i, p_j must be the zero polynomial. $\Box$

The proof given above shows that A_Δ is a free module over the ring $\mathbf{R}[\theta_1, \ldots, \theta_d]$ and that $\eta_1, \ldots, \eta_m$ is a monomial basis. Further, there are exactly $h_i(\Delta)$ elements of the basis of degree i. We can construct another monomial basis in the following way. Order the monomials lexicographically by defining $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if $a_1 = b_1, \ldots, a_j = b_j$ but $a_{j+1} < b_{j+1}$. Now choose a basis in a greedy fashion by letting $\eta_1 := 1$ and η_j be the first monomial lexicographically that you cannot represent using $\eta_1, \ldots, \eta_{j-1}$. Call the resulting basis M. It will still have exactly $h_i(\Delta)$ elements of degree i.

Theorem 8.4 *The basis M is an order ideal of monomials.*

Proof. We need to show that if η is in M then so are all its divisors. For suppose not. Then there is a divisor m of η that was not chosen. It was considered before η because $m < \eta$ ($m = m'\eta_i < m'm = \eta$). It was rejected because $m = \sum p_i \eta_i$ for the η_i in M that are less than m. But $\eta = mm'$ for some monomial m'. So $\eta = \sum p_i m' \eta_i$. But $m'\eta_i < \eta$ for each i, so each of these can be expressed in terms of the η_j in M that are less than η. Hence η itself can be expressed in terms of the preceding η_j in M, contradicting the fact that η is a basis element. $\square$

Corollary 8.5 *If Δ is a Cohen-Macaulay simplicial $(d-1)$-complex with n vertices, then*

$$h_i \leq \binom{n-d+i-1}{i}, \quad i = 1, \ldots, d.$$

Proof. There are precisely $h_1(\Delta) = n - d$ monomials of degree one in M. So by Exercise 3.10 there can be no more than $\binom{n-d+i-1}{i}$ monomials of degree i in M. Therefore $h_i \leq \binom{n-d+i-1}{i}$. $\square$

This provides a new proof of the Upper Bound Theorem for simplicial d-polytopes. As mentioned above, triangulated $(d-1)$-spheres S are also Cohen-Macaulay. It is a simpler fact to prove that the Dehn-Sommerville equations are also satisfied. Using Theorem 8.2 one can show that there must be $h_i(S)$ monomials of degree i in a monomial basis for B. This is done by realizing that a basis for A as a vector space over $\mathbf{R}$ is obtained by multiplying monomials in the θ_i by elements in the basis M for B. From this one immediately has

Theorem 8.6 (Upper Bound Theorem for Spheres, Stanley) *Let S be a triangulated $(d-1)$-sphere with n vertices. Then $h_i(S) \leq h_i(C(n,d))$, $i = 0, \ldots, d$, and $f_j(S) \leq f_j(C(n,d))$, $j = 0, \ldots, d-1$.*

8.3 The g-Theorem

McMullen [McM71] in 1971 (preceding Stanley's results on Cohen-Macaulay complexes) conjectured a set of conditions to completely characterize the f-vectors of simplicial convex polytopes. These conditions are often referred to as *McMullen's conditions*, and their ultimate verification as the *g-Theorem.*

Billera and Lee [BL80, BL81] constructed polytopes to establish the sufficiency of McMullen's conditions in 1980, and shortly after Stanley [Sta80] invoked some deep results in algebraic geometry to confirm their necessity.

Theorem 8.7 (g-Theorem, Billera-Lee-Stanley) $(h_0, \ldots, h_d) \in \mathbf{Z}_+^{d+1}$ *is the h-vector of a simplicial convex d-polytope iff*

1. $h_i = h_{d-i}$, $i = 0, \ldots, \lfloor d/2 \rfloor$, *(Dehn-Sommerville Equations) and*
2. $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-sequence, where $g_0 = h_0$ and $g_i = h_i - h_{i-1}$, $i = 1, \ldots, \lfloor d/2 \rfloor$.*

PROOF.

Sketch of sufficiency: Let us consider an example where $d = 6$, $f = (1, 10, 43, 102, 141, 108, 36)$, and $h = (1, 4, 8, 10, 8, 4, 1)$, $g = (1, 3, 4, 2)$. Consider the cyclic polytope $C(n, d+1)$ where $n = f_0 = h_1 + d$. List those facets which contain 1 and have even cardinality right end set. Except possibly for vertex 1, the vertices of the facets fall naturally into pairs. Let V^i be the set of facets such that exactly i of these pairs are not in their "leftmost possible position." For all i choose the first g_i sets in V^i, using reverse lexicographic order. These are the facets of a simplicial d-ball Δ, and they are shellable in reverse lexicographic order. Further, if a chosen F is in V^i then it contributes to $h_i(\Delta)$ during the shelling.

	1	2	3	4	5	6	7	8	9	10
0*	1	2	3	4	5	6	7			
1*	1	2	3	4	5		7	8		
2*	1	2	3		5	6	7	8		
3*	1		3	4	5	6	7	8		
1*	1	2	3	4	5			8	9	
2*	1	2	3		5	6		8	9	
3*	1		3	4	5	6		8	9	
2*	1	2	3			6	7	8	9	
3	1		3	4		6	7	8	9	
3	1			4	5	6	7	8	9	
1*	1	2	3	4	5				9	10
2*	1	2	3		5	6			9	10
3	1		3	4	5	6			9	10
2	1	2	3			6	7		9	10
3	1		3	4		6	7		9	10
3	1			4	5	6	7		9	10
2	1	2	3				7	8	9	10
3	1		3	4			7	8	9	10
3	1			4	5		7	8	9	10
3	1				5	6	7	8	9	10

So $h(\Delta) = (1,3,4,2,0,0,0,0)$. The next step is to find a point z from which the facets of $C(n,d+1)$ that are visible are precisely those in Δ, and such that z is beneath the remaining facets of $C(n,d+1)$ (this is not easy!). Let Q be the simplicial $(d+1)$-polytope $\operatorname{conv}(C(n,d+1) \cup \{z\})$ and let P be the simplicial d-polytope that is a vertex figure at z (slice off z with a hyperplane and consider the intersection of Q with the hyperplane). Then P will have the desired h-vector. To see this, notice that the faces of P are in one-to-one correspondence with the boundary faces of Δ. Let Σ be the simplicial d-sphere $\Sigma = \Delta \cup (\partial\Delta \cdot z)$ (we are joining the boundary faces of Δ to the new point z). Then $f_j(\Delta) + f_{j-1}(\partial\Delta) = f_j(\Sigma)$ and the same will be true for the h-vectors. So we have

$$\begin{array}{c|cccccccc|l} & 1 & 3 & 4 & 2 & 0 & 0 & 0 & 0 & h(\Delta) \\ + & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h(\partial\Delta) = h(P) \\ \hline = & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h(\Sigma) \end{array}$$

But both $\partial\Delta$ and Σ, being spheres, satisfy the Dehn-Sommerville equations (are symmetric). Thus we are forced to fill in the above numbers in the following way (see Theorem 3.24 and

Example 3.25):

$$\begin{array}{rcccccccccl}
 & 1 & 3 & 4 & 2 & 0 & 0 & 0 & 0 & & h(\Delta) \\
+ & & 1 & 4 & 8 & 10 & 8 & 4 & 1 & & h(\partial\Delta) = h(P) \\
\hline
= & 1 & 4 & 8 & 10 & 10 & 8 & 4 & 1 & & h(\Sigma)
\end{array}$$

Sketch of necessity: Proceed as in the proof that shellable complexes are Cohen-Macaulay. Prove that for a particular choice of θ_i there exists an element $\omega \in B_1$ such that multiplication by ω^{d-2i} is a bijection from B_i to B_{d-i}, $i = 0, \ldots, \lfloor d/2 \rfloor$. The Dehn-Sommerville equations are an immediate consequence, but there is more. Let $C = B/(\omega) = C_0 \oplus C_1 \oplus \cdots \oplus C_{\lfloor d/2 \rfloor}$. Then $\dim C_i = \dim B_i - \dim B_{i-1} = h_i - h_{i-1} = g_i$, $i = 1, \ldots, \lfloor d/2 \rfloor$. There exists a basis for C as a vector space over $\mathbf{R}$ that is an order ideal of monomials, so $(g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$ is an M-sequence.

The existence of ω is proved by associating with P a certain complex projective variety X_P, whose cohomology ring is isomorphic to B. The Hard Lefschetz Theorem implies the existence of the desired ω. □

There are partial extensions of Stanley's result above to nonsimplicial convex polytopes. There are also extensions of shifting techniques, due to Björner and Kalai [BK88], allowing the characterization of f-Betti vector pairs for simplicial complexes.

9 Generalized Stress

Refer to the paper by Lee [Lee96], in which a generalization of the notion of stress to higher dimensional faces is discussed.

9.1 Linear Subspaces and the Grassmann-Plücker Relations

The material in this section was taken from J. Bokowski and B. Sturmfels, *Computational Synthetic Geometry*. The goal of this section is to "encode" or "coordinatize" linear subspaces of $\mathbf{R}^d$ suitably. For example, suppose S is a 1-dimensional linear subspace of $\mathbf{R}^d$. Then S is completely determined by any one specific nonzero element x of S, since all other elements are multiples of x. Another way of saying this is that every basis vector of S is projectively equivalent to x (i.e., a nonzero multiple of x). What is the appropriate generalization if S is higher dimensional?

Definition 9.1 For $0 \leq n \leq d$, define $\Lambda(d,n) = \{(\ell_1,\ldots,\ell_n) : 1 \leq \ell_1 < \cdots < \ell_n \leq d\}$, the set of all increasing ordered n-tuples from $\{1,\ldots,d\}$. For $n \times d$ matrix A and $\ell = (\ell_1,\ldots,\ell_n) \in \Lambda(d,n)$, define A_ℓ to be the $n \times n$ submatrix of A obtained by selecting only the columns of A indexed by ℓ. I.e., denoting column i of A by a^i, $A_\ell = [a^{\ell_1},\ldots,a^{\ell_n}]$.

Definition 9.2 Let S be a linear subspace of $\mathbf{R}^d$ of dimension n. Choose a basis for S and list these vectors as the rows of an $n \times d$ matrix A. Calculate all $n \times n$ subdeterminants of A and list these numbers as the coordinates of a vector $(\det A_\ell)_{\ell \in \Lambda(d,n)} \in \mathbf{R}^{\binom{d}{n}}$. This vector is the *Plücker vector* for S, and its components are the *Plücker coordinates* for S.

Theorem 9.3 *If the above procedure is carried out with two different bases of S, the resulting Plücker vectors will be nonzero multiples of each other.*

Proof. Suppose A' is another $n \times d$ matrix whose rows form a basis for S. Then there exists a nonsingular $n \times n$ matrix B such that $A' = BA$ (why?). So $\det A'_\ell = \det B \det A_\ell$ for all $\ell \in \Lambda(d,n)$. Therefore the Plücker vector derived from A' equals $\det B$ multiplied by the Plücker vector derived from A. $\square$

Exercise 9.4 Calculate the Plücker vector for the linear subspace $S = \{x \in \mathbf{R}^4 : Mx = O\}$, where

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{bmatrix}$$

Exercise 9.5 Determine a basis for the 2-dimensional linear subspace of $\mathbf{R}^4$ whose Plücker vector is

$$\begin{pmatrix} \det A_{(1,2)} & \det A_{(1,3)} & \det A_{(1,4)} & \det A_{(2,3)} & \det A_{(2,4)} & \det A_{(3,4)} \\ 2 & 6 & 2 & 5 & 4 & 7 \end{pmatrix}$$

Exercise 9.6 Let S be a linear subspace of $\mathbf{R}^d$. Define $S^\perp$ to be the linear subspace $\{x \in \mathbf{R}^d : x^T y = 0 \ \forall y \in S\}$.

1. Suppose the rows of the matrix $[I, M]$ form a basis for S. Show that the rows of the matrix $[-M^T, I]$ form a basis for $S^\perp$.

2. What is the relationship between the Plücker vectors for S and $S^\perp$?

Theorem 9.3 shows that the Plücker vector of an n-dimensional linear subspace is projectively unique and can be regarded as a point in projective space $\mathbf{P}^{\binom{d}{n}}$. The collection of all such points is called the *Grassmannian.* What this collection look like? Can any vector in $\mathbf{R}^{\binom{d}{n}}$ be the Plücker vector of some n-dimensional subspace of $\mathbf{R}^d$? The next result shows that this is not the case.

Theorem 9.7 (Grassmann-Plücker Relations) *Suppose A is any $n \times d$ matrix, ℓ is any member of $\Lambda(d, n+1)$, and m is any member of $\Lambda(d, n-1)$. Then*

$$\sum_{i=1}^{n+1} (-1)^i \det\left[a^{\ell_1}, \ldots, a^{\ell_{i-1}}, a^{\ell_{i+1}}, \ldots, a^{\ell_{n+1}}\right] \det\left[a^{\ell_i}, a^{m_1}, \ldots, a^{m_{n-1}}\right] = 0.$$

Before proving this theorem, we need to recall some properties of determinants.

Theorem 9.8

1. *If a square matrix has a column of zeros, then its determinant is zero. Similarly for rows.*

2. *If a square matrix has two identical columns, then its determinant is zero. Similarly for rows.*

3. *If $[x^1, \ldots, x^n]$ is a square matrix and $\alpha \in \mathbf{R}$, then*

$$\det\left[\alpha x^1, x^2, \ldots, x^n\right] = \alpha \det\left[x^1, x^2, \ldots, x^n\right].$$

   Similarly for rows.

4. *If* $[x^1+\overline{x}^1, x^2, \ldots, x^n]$ *is a square matrix, then*

$$\det\left[x^1+\overline{x}^1, x^2, \ldots, x^n\right] = \det\left[x^1, x^2, \ldots, x^n\right] + \det\left[\overline{x}^1, x^2, \ldots, x^n\right].$$

Similarly for rows.

5. *If* $[x^1, \ldots, x^n]$ *is a square matrix, then*

$$\det\left[x^{\pi(1)}, \ldots, x^{\pi(n)}\right] = (\operatorname{sign}\pi)\det\left[x^1, \ldots, x^n\right]$$

where π *is a permutation of* $\{1,\ldots,n\}$ *and* $\operatorname{sign}\pi$ *is the* sign *of* π*; i.e.,* $(-1)^t$*, where* t *is the number of transpositions of adjacent elements needed to transform* $(1,\ldots,n)$ *into* $(\pi(1),\ldots,\pi(n))$*. Similarly for rows.*

PROOF. Exercise. □

Theorem 9.9 *Suppose* $[x^1, \ldots, x^{n+1}]$ *is an* $n\times(n+1)$ *matrix. Then*

$$\sum_{i=1}^{n+1}(-1)^i\det\left[x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^{n+1}\right]x^i = O.$$

PROOF. Fix $1\le j\le n$. Place a copy of row j at the top of the matrix. Then

$$\det\begin{bmatrix} x^1_j & \cdots & x^{n+1}_j \\ x^1 & \cdots & x^{n+1}\end{bmatrix} = 0$$

since two rows are identical. Expanding this determinant along the first row,

$$\sum_{i=1}^{n+1}(-1)^i\det\left[x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^{n+1}\right]x^i_j = 0.$$

The result now follows since this is true for all j. □

PROOF OF THEOREM 9.7.

$$\begin{aligned}
0 &= \det\left[O, a^{m_1},\ldots,a^{m_{n-1}}\right] \\
&= \det\left[\left(\sum_{i=1}^{n+1}(-1)^i\det\left[a^{\ell_1},\ldots,a^{\ell_{i-1}},a^{\ell_{i+1}},\ldots,a^{\ell_{n+1}}\right]a^{\ell_i}\right), a^{m_1},\ldots,a^{m_{n-1}}\right] \\
&= \sum_{i=1}^{n+1}(-1)^i\det\left[a^{\ell_1},\ldots,a^{\ell_{i-1}},a^{\ell_{i+1}},\ldots,a^{\ell_{n+1}}\right]\det\left[a^{\ell_i},a^{m_1},\ldots,a^{m_{n-1}}\right]. \ \square
\end{aligned}$$

Exercise 9.10 Let $A = [a^1, \ldots, a^d]$ be an $n \times d$ matrix. For $B = \{\ell_1, \ldots, \ell_n\} \subseteq \{1, \ldots, d\}$, define B to be a *basis* (with respect to A) if $\det[a^{\ell_1}, \ldots, a^{\ell_n}] \neq 0$. Use Theorem 9.7 to prove the *basis exchange property*: If B and B' are two bases and $k \in B$, then there exists $k' \in B'$ such that $(B \setminus \{k\}) \cup \{k'\}$ is a basis.

The last result shows that the Grassmann-Plücker relations are both necessary and sufficient to characterize Plücker vectors.

Theorem 9.11 *A nonzero vector $v = (v_\ell)_{\ell \in \Lambda(d,n)} \in \mathbf{R}^{\binom{d}{n}}$ is the Plücker vector of some n-dimensional linear subspace of $\mathbf{R}^d$ iff*

$$\sum_{i=1}^{n+1} (-1)^i v_{(\ell_1,\ldots,\ell_{i-1},\ell_{i+1},\ldots,\ell_{n+1})} v_{(\ell_i,m_1,\ldots,m_{n-1})} = 0$$

for all $\ell \in \Lambda(d, n+1), m \in \Lambda(d, n-1)$, where we extend the definition of v_k to all n-tuples $k = (k_1, \ldots, k_n)$ drawn from $\{1, \ldots, d\}$ by defining $v_k = 0$ if $k_i = k_j$ for any $i \neq j$, and by defining $v_{(k_{\pi(1)},\ldots,k_{\pi(n)})}$ to be $(\operatorname{sign} \pi) v_{(k_1,\ldots,k_n)}$ for any permutation π of $\{1, \ldots, n\}$.

Proof. Without loss of generality assume $v_{(1,\ldots,n)} \neq 0$. Choose any $a^1, \ldots, a^n \in \mathbf{R}^n$ such that $\det[a^1, \ldots, a^n] = v_{(1,\ldots,n)}$. For $j = n+1, \ldots, d$, define

$$a^j = \frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} v_{(1,\ldots,i-1,j,i+1,\ldots,n)} a^i.$$

We will prove by induction on k that

$$v_{(\ell_1,\ldots,\ell_n)} = \det\left[a^{\ell_1}, \ldots, a^{\ell_n}\right]$$

for all $\ell \in \Lambda(k, n)$ where $n \leq k \leq d$. This is true for $k = n$ by construction, so suppose $k > n$ and assume that the above formula holds for all $\ell \in \Lambda(k-1, n)$. Let $\ell \in \Lambda(k, n) \setminus \Lambda(k-1, n)$;

i.e., $\ell_n = k$. Then

$$\begin{aligned}
&\det\left[a^{\ell_1}, \ldots, a^{\ell_{n-1}}, a^k\right] \\
&= \det\left[a^{\ell_1}, \ldots, a^{\ell_{n-1}}, \left(\frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} v_{(1,\ldots,i-1,k,i+1,\ldots,n)} a^i\right)\right] \\
&= \frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} v_{(1,\ldots,i-1,k,i+1,\ldots,n)} \det\left[a^{\ell_1}, \ldots, a^{\ell_{n-1}}, a^i\right] \\
&= \frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} v_{(1,\ldots,.i-1,k,i+1,\ldots,n)} v_{(\ell_1,\ldots,\ell_{n-1},i)} \\
&= \frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} (-1)^{n-i} v_{(1,\ldots,i-1,i+1,\ldots,n,k)} (-1)^{n-1} v_{(i,\ell_1,\ldots,\ell_{n-1})} \\
&= -\frac{1}{v_{(1,\ldots,n)}} \sum_{i=1}^{n} (-1)^{i} v_{(1,\ldots,i-1,i+1,\ldots,n,k)} v_{(i,\ell_1,\ldots,\ell_{n-1})} \\
&= \frac{1}{v_{(1,\ldots,n)}} (-1)^{n+1} v_{(1,\ldots,n)} v_{(k,\ell_1,\ldots,\ell_{n-1})} \\
&= v_{(\ell_1,\ldots,\ell_{n-1},k)}. \ \square
\end{aligned}$$

10 Minkowski's Theorem

Some of the material in this section is taken from Grünbaum [Grü67].

Theorem 10.1 (Minkowski) *Let $P \subset \mathbf{R}^d$ be a convex d-polytope with facets $F_1, \ldots, F_n$. Let $u^1, \ldots, u^n$ be the respective outer unit normals of the facets. Let V_i be the $(d-1)$-dimensional volume of the facet F_i, $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} V_i u^i = 0.$$

PROOF. Let c be any point in the interior of P and let d_i be the distance of facet F_i from c, $i = 1, \ldots, n$. Let V be the volume of P. Then

$$V = \frac{1}{d} \sum_{i=1}^{n} d_i V_i.$$

Let the equation of the supporting hyperplane to facet F_i be $u^i \cdot x = b_i$, $i = 1, \ldots, n$. Then

$$d_i = \frac{|u^i \cdot c - b_i|}{\|u^i\|} = b_i - u^i \cdot c$$

so

$$V = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c) V_i.$$

Let $t > 0$ be small enough so that the ball of radius t centered at c lies within the interior of P. Let u be any unit vector and consider the point $c' = c + tu$. Then computing the volume of P from c' we have

$$V = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c') V_i = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c - u^i \cdot tu) V_i.$$

Subtracting the two expressions for V gives

$$0 = \frac{1}{d} \sum_{i=1}^{n} (u^i \cdot tu) V_i$$

so

$$u \cdot \sum_{i=1}^{n} V_i u^i = 0$$

for all unit vectors u. This implies that the sum must itself be the zero vector. $\square$

Theorem 10.2 (Minkowski) *Let $v^1, \ldots, v^n$ be vectors in $\mathbf{R}^d$ such that*

1. *$v^1, \ldots, v^n$ span $\mathbf{R}^d$,*

2. *No v^i is a positive multiple of any other v^j,*

3. *$\sum_{i=1}^{n} v^i = 0$.*

Then there exists a convex d-polytope P with facets F_i, $i = 1, \ldots, n$, such that the unit outer normals are $u^i = v^i/\|v^i\|$ and the $(d-1)$-dimensional volumes are $V_i = \|v^i\|$, respectively.

Proof. List the vectors v^i as columns of a matrix A. Note that $Ae = 0$, where $e \in \mathbf{R}^n$ is the vector $(1, \ldots, 1)^T$. For $b \in \mathbf{R}^n$ define the polyhedron $P(b) = \{x \in \mathbf{R}^d : A^T x \leq b\}$. Then let $B = \{b \in \mathbf{R}^n : P(b) \neq \emptyset, Ab = 0, \text{ and } e^T b = 1\}$.

Claim 1. B is a convex polyhedron, since it is a projection of $\{(b, x) \in \mathbf{R}^{d+n} : A^T x - b \leq 0, Ab = 0, \text{ and } e^T b = 1\}$.

Claim 2. B is bounded, and hence a convex polytope. For choose any direction $c \neq 0$ such that $Ac = 0$ and $e^T c = 0$. Let $b \in B$ and consider the ray $b + tc$, $t > 0$. We need to show that if t is large enough, then this ray is not in B. Consider the following dual pair of linear programs.

$$\begin{array}{cc} \max 0^T x & \min (b+tc)^T y \\ A^T x \leq b + tc & Ay = 0 \\ & y \geq 0 \\ & \\ (I) & (II) \end{array}$$

We need to show that (I) is not always feasible. But (I) is feasible iff (II) (which is clearly feasible) has bounded objective function value. This is equivalent to the nonexistence of y such that $Ay = 0$, $y \geq 0$, and $(b + tc)^T y < 0$. Choose $\varepsilon > 0$ and let $y = e - \varepsilon c$. Make ε small enough so that $y \geq 0$. Then $Ay = Ae - \varepsilon Ac = 0 - 0 = 0$. But $(b + tc)^T y = b^T e - \varepsilon b^T c + tc^T e - \varepsilon tc^T c = b^T e - \varepsilon b^T c - \varepsilon t \|c\|^2$ which is negative when t is sufficiently large.

Claim 3. $P(b)$ is bounded for all $b \in B$, hence has finite volume. For assume $P(b)$ is not bounded. Then there is some $x \in P(b)$ and some direction $z \neq 0$ such that $A^T(x + tz) \leq b$ for all $t \geq 0$. So $A^T z \leq 0$. If there is strict inequality anywhere, then $0 = z^T(Ae) = (z^T A)e < 0$, a contradiction. So $A^T z = 0$. But the columns of A span $\mathbf{R}^d$, so z, being orthogonal to all of the columns, must itself be the zero vector, a contradiction.

From our claims we now know that $V(P(b))$, the volume of $P(b)$, is well-defined for all $b \in B$. Also, since B is closed and bounded and $V(P(b))$ is a continuous function of b, we can consider the problem $\max\{V(P(b)) : b \in B\}$. Note that e/n is in B and that

$P(e/n)$ contains the origin in its interior since $e/n > 0$, so the maximization problem has a positive maximum. The maximum is achieved by some $b^* \in B$. Let $P^* = P(b^*)$. Let $F_1, \ldots, F_n$ be its facets, with unit outer normals $u_1, \ldots, u_n$, and $(d-1)$-volumes $V_1, \ldots, V_n$, respectively. For $h \in \mathbf{R}^n$ consider the function $V(h) = V(P(b^* + h))$. The gradient of $V(h)$ at $h = 0$ is $(V_1/\|v^1\|, \ldots, V_n/\|v^n\|)$ (remembering that no two v^i are positive multiples of each other). Choose any $c \in \mathbf{R}^n$ such that $Ac = 0$ and $e^T c = 0$ and consider the function $V(t) = V(P(b^* + tc))$. Since P^* is optimal, we have $dV/dt = 0$ so

$$\sum_{i=1}^{n} \frac{V_i}{\|v^i\|} c_i = 0.$$

Also, by Theorem 10.1,

$$\sum_{i=1}^{n} V_i \frac{v^i}{\|v^i\|} = 0.$$

So the vector $(V_1/\|v^1\|, \ldots, V_n/\|v^n\|)$ is orthogonal to all of the rows of A and is orthogonal to all affine relations on the columns of A. But then this vector must be a multiple of e. So there is some positive number k such that $V_i = k\|v^i\|$, $i = 1, \ldots, n$. Scale P^*, if necessary, to obtain the desired polytope. □

We omit the proof of the following stronger result that states that a polytope is essentially uniquely determined by its unit facet normals and facet volumes.

Theorem 10.3 (Minkowski) *The polytope which exists by the previous theorem is unique up to translation.*

There are analogs of these theorems for continuously curved convex bodies in which a curvature function plays the role of the facet volumes. See Bonnesen-Fenchel [BF87].

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