Math 603, Spring 2003, HW 2, due 2/10/2003

Part A

AI) (Topologies and Presheaves). Let X be a topological space. We can make a category, $\mathcal{T}_X$, which is specified by and specifies the topology as follows: $\mathcal{O}\mathrm{b}\,\mathcal{T}_X$ consists of the open sets in X. If $U, V \in \mathcal{O}\mathrm{b}\,\mathcal{T}_X$, we let

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset & \text{if } U \not\subseteq V, \\ \{\mathrm{incl}\} & \text{if } U \subseteq V, \end{cases}$$

here $\{\mathrm{incl}\}$ is the one element set consisting of the inclusion map $\mathrm{incl} : U \to V$.

(a) Show that $U \prod_X V$—the fibred product of U and V (over X) in $\mathcal{T}_X$—is just $U \cap V$. Therefore $\mathcal{T}_X$ has finite fibred products.

(b) If $\mathcal{C}$ is a given category (think of $\mathcal{C}$ as $\mathcal{S}$ets, $\mathcal{A}$b, or more generally Λ-$\mathcal{M}$odules) a *presheaf on X with values in $\mathcal{C}$* is a cofunctor from $\mathcal{T}_X$ to $\mathcal{C}$. So, F is a presheaf iff $(\forall \text{ open } U \subseteq X)(F(U) \in \mathcal{C})$ and if $U \hookrightarrow V$, we have a map $\rho_V^U : F(V) \to F(U)$ (in $\mathcal{C}$) usually called *restriction from V to U*. Of course, we assume $\rho_V^W = \rho_U^W \circ \rho_V^U$. The basic example, from which all the terminology comes, is this:

$$\mathcal{C} = \mathbb{R}\text{-modules } (= \text{vector spaces over } \mathbb{R})$$
$$F(U) = \{\text{continuous real valued functions on open set } U\}.$$

Now recall that a category is an *abelian category* iff for each morphism $A \xrightarrow{\varphi} B$ in $\mathcal{C}$, there are two pairs: $(\ker\varphi, i)$ and $(\mathrm{coker}\,\varphi, j)$ with $\ker\varphi$ and $\mathrm{coker}\,\varphi$ objects of $\mathcal{C}$ and $i : \ker\varphi \to A$, $j : B \to \mathrm{coker}\,\varphi$ so that:

1) $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a group, abelian, operation denoted $+$
2) $\ker\varphi \to A \to B$ is zero in $\mathrm{Hom}_{\mathcal{C}}(\ker\varphi, B)$
3) If $C \xrightarrow{u} A \to B$ is zero, $\exists$ a unique morphism $C \to \ker\varphi$ so that u is the composition $C \to \ker\varphi \xrightarrow{i} A$
4) Similar to 3) for coker, with appropriate changes.

Define $\mathrm{Im}\,\varphi$ as $\ker(B \xrightarrow{j} \mathrm{coker}\,\varphi)$. Now exact sequences make sense in $\mathcal{C}$ (easy, as you see). Write $\mathcal{P}(X,\mathcal{C})$ for the category of presheaves on X with values in $\mathcal{C}$. If $\mathcal{C}$ is abelian show that $\mathcal{P}(X,\mathcal{C})$ is an abelian category, too, in a natural way.

(c) If $A \in \mathcal{O}\mathrm{b}\,\mathcal{C}$, we can make a presheaf $\underline{A}$ by: $\underline{A}(U) = A$, all open U and $V \hookrightarrow U$ then $\rho_V^U = id_A$. This is the *constant presheaf* with values in A. Generalize it as follows: Fix open U of X, define $\underline{A}_U$ by:

$$\underline{A}_U(W) = \coprod_{\mathrm{Hom}(W,U)} A = \begin{cases} (0) & \text{if } W \not\subseteq U \\ A & \text{if } W \subseteq U. \end{cases}$$

Show $\underline{A}_U$ is a presheaf and $\underline{A}$ is one of these $\underline{A}_U$; which one? Generalize further: Say $\mathcal{F}$ is a presheaf of sets on X, define $\underline{A}_{\mathcal{F}}$ by:

$$\underline{A}_{\mathcal{F}}(W) = \coprod_{\mathcal{F}(W)} A = \big\{\text{functions} : \mathcal{F}(W) \to A \mid \text{these functions have finite support}\big\}.$$

Make $\underline{A}_{\mathcal{F}}$ into a presheaf on X; it is a clear generalization of $\underline{A}_U$ and this, in turn, generalizes $\underline{A}$.

(d) Just as with the defining example in (b), which is called the *presheaf of germs of continuous functions on X* so we can define the presheaf of germs of C^k-functions, real-analytic functions, complex holomorphic functions, meromorphic functions when X is a real (resp. complex) manifold. Namely:

$$\begin{aligned} C^k(U) &= \{f : U \to \mathbb{R} \mid f \text{ is } C^k \text{ on } U\}. \quad 0 \le k \le \infty \\ C^{\infty}(U) &= \{f : U \to \mathbb{R} \mid f \text{ is real analytic on } U\} \\ \mathrm{Hol}(U) &= \{f : U \to \mathbb{C} \mid f \text{ is holomorphic on } U\} \\ \mathrm{Mer}(U) &= \{f : U \to \mathbb{C} \mid f \text{ is meromorphic on } U\}. \end{aligned}$$

Prove: The collection $\{\underline{\mathbb{Z}}_U \mid U \text{ open in } X\}$ is *a set of generators* for $\mathcal{P}(X, \mathcal{A}\mathrm{b})$; that is: $\forall$ presheaves $\mathcal{F}$, $\exists$ subcollection of the U's, say $\{U_\alpha \mid \alpha \in \Lambda\}$, so that there is a surjection $\coprod_I \left(\coprod_{\alpha\in\Lambda} \underline{\mathbb{Z}}_U \right) \twoheadrightarrow \mathcal{F}$, for some *set* I. (Then it turns out that every presheaf embeds in an injective presheaf.)

(e) Now sheaves are special kinds of presheaves. Say $U \in \mathcal{T}_X$ and we have a family of morphisms of $\mathcal{T}_X$: $\{U_\alpha \to U\}_{\alpha\in\Lambda}$ (we'll suppress mention of Λ in what follows). We call this family a *covering family* $\iff \bigcup_\alpha U_\alpha = U$, i.e. the U_α are an open covering of U. Of course, if $\xi \in F(U)$, then $\rho_U^{U_\alpha} \in F\big(U_\alpha\big)$, each α; here, F is a presheaf. Hence we get a map

$$\theta : F(U) \to \prod_\alpha F\big(U_\alpha\big).$$

Now if $\xi_\alpha \in F\big(U_\alpha\big)$, for each α, then $\rho_{U_\alpha}^{U_\alpha\cap U_\beta}\big(\xi_\alpha\big)$ lies in $F\big(U_\alpha \cap U_\beta\big)$ therefore we get a map

$$p_{1,\alpha} : F\big(U_\alpha\big) \to \prod_\beta F\big(U_\alpha \cap U_\beta\big).$$

Take the product of these over α and get a map

$$p_1 : \prod_\alpha F\big(U_\alpha\big) \to \prod_{\alpha,\beta} F\big(U_\alpha \cap U_\beta\big).$$

If $\xi_\beta \in F\big(U_\beta\big)$ then $\rho_{U_\beta}^{U_\alpha\cap U_\beta}\big(\xi_\beta\big) \in F\big(U_\alpha \cap U_\beta\big)$ therefore we get a map

$$p_{2,\beta} : F\big(U_\beta\big) \to \prod_\alpha F\big(U_\alpha \cap U_\beta\big).$$

Again the product over β gives:

$$p_2 : \prod_\beta F\big(U_\beta\big) \to \prod_{\alpha,\beta} F\big(U_\alpha \cap U_\beta\big),$$

hence we get two maps:

$$\prod_\gamma F\big(U_\gamma\big) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{\alpha,\beta} F\big(U_\alpha \cap U_\beta\big).$$

The definition of a *sheaf* is: a *sheaf, F, of sets* is a presheaf, F, of sets so that ($\forall$ open U) $\Big(\forall \text{ covers } \big\{U_\alpha \to U\big\}_\alpha\Big)$ the sequence

$$F(U) \xrightarrow{\theta} \prod_\gamma F\big(U_\gamma\big) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{\alpha,\beta} F\big(U_\alpha \cap U_\beta\big) \qquad \text{(S)}$$

is exact in the sense that θ *maps* $F(U)$ *bijectively* to the set $\big(\xi_\gamma\big) \in \prod_\gamma F\big(U_\gamma\big)$ for which $p_1\big(\big(\xi_\gamma\big)\big) = p_2\big(\big(\xi_\gamma\big)\big)$. Show that the presheaves of germs of continuous, k-fold continuous, differentiable, analytic, holomorphic and meromorphic functions are all sheaves. In so doing understand what exactness of sequence (S) means.

Prove, however, that $\underline{A}$ is NOT a sheaf. (Note: a sheaf with values in $\mathcal{A}$b or RNG or Ω-groups is just a presheaf with these values which forms a sheaf of sets.) For which presheaves, $\mathcal{F}$, is $\underline{A}_{\mathcal{F}}$ a sheaf?

AII) Let k be a field, X an indeterminate (or transcendental) over k. Write $A = k[X]$ and consider an ideal, $\mathfrak{a}$, of A. The ideal $\mathfrak{a}$ determines a topology on $k[X]$—called the $\mathfrak{a}$-*adic topology*—defined by taking as a fundamental system of neighborhoods of 0 the powers $\{\mathfrak{a}^n \mid n \geq 0\}$ of $\mathfrak{a}$. Then a fundamental system of neighborhoods at $\xi \in A$ is just the collection $\{\xi + \mathfrak{a}^n \mid n \geq 0\}$.

(a) Show A becomes a topological ring (i.e. addition and multiplication are continuous) in this topology. When is A Hausdorff in this topology?

(b) The rings $A/\mathfrak{a}^n = A_n$ form a left mapping system. Write

$$\widehat{A} = \varprojlim_{n} A/\mathfrak{a}^n$$

and call $\widehat{A}$ the $\mathfrak{a}$-*adic completion of* A. There is a map $A \to \widehat{A}$; when is it injective?

(c) Consider $\mathfrak{a} = (X) =$ all polynomials with no constant term. The ring $\widehat{A}$ in this case has special notation: $k[[X]]$. Establish an isomorphism of $k[[X]]$ with the *ring of formal power series over* k (in X) i.e. with the ring consisting of sequences $\big(c_n\big)$, $n \geq 0$, $c_n \in k$ with addition and multiplication defined by:

$$\big(c_n\big) + \big(d_n\big) = \big(c_n + d_n\big)$$
$$\big(c_n\big) \cdot \big(d_n\big) = \big(e_n\big), \quad e_n = \sum_{i+j=n} c_i d_j$$

$\left(\big(c_n\big) \leftrightarrow \sum_{n=0}^{\infty} c_n X^n \text{ explains the name}\right)$

(d) Show $k[X] \hookrightarrow k[[X]]$, that $k[[X]]$ is an integral domain and a local ring. What is its maximal ideal? Now $(X) = \mathfrak{a}$ is a prime ideal of $k[X]$, so we can form $k[X]_{(X)}$. Prove that

$$k[X] \subseteq k[X]_{(X)} \subseteq k[[X]].$$

We have the (prime) ideal $(X)^e$ of $k[X]_{(X)}$. Form the completion of $k[X]_{(X)}$ with respect to the $(X)^e$-adic topology. What ring do you get?

AIII) Prove that in the category of commutative A-algebras, the tensor product is the coproduct:

$$B \otimes_A C \cong B \coprod_A C.$$

Which A-algebra is the product $B \prod C$ (in commutative A-algebras)?

Part B

BI) Here A is a commutative ring and we write $M_n(A)$ for the ring of $n \times n$ matrices over A.

(a) Prove: the following are equivalent

1) A is noetherian
2) For some n, $M_n(A)$ has the ACC on 2-sided ideals
3) For all n, $M_n(A)$ has the ACC on 2-sided ideals.

(b) Is this still valid if "noetherian" is replaced by "artinian" and "ACC" by "DCC"? Proof or counterexample.

(c) Can you make this quantitative? For example, suppose all ideals of A are generated by less than or equal to N elements. What can you say about an upper bound for the number of generators of the ideals of $M_n(A)$? How about the converse?

BII) Refer to problem AII. Write $k((X))$ for $\mathrm{Frac}(k[[X]])$.

(a) Show that

$$k((X)) = \left\{ \sum_{j=-\infty}^{\infty} a_j X^j \mid a_j \in k, \text{ all } j \text{ and } (\exists\, N)(a_j = 0 \text{ if } j < N) \right\}$$

where on the right hand side we use the obvious addition and multiplication for such expressions. If $\xi \in k((X))$, write $\mathrm{ord}(\xi) = N \iff N =$ largest integer so that $a = 0$ when $j < N$; here, $\xi \neq 0$. If $\xi = 0$, set $\mathrm{ord}(\xi) = \infty$. One sees immediately that $k[[X]] = \{\xi \in k[[X]] \mid \mathrm{ord}(\xi) \geq 0\}$.

(b) Write $\mathcal{U}$ for $\mathbb{G}_m(k[[X]])$ and $\mathcal{M}$ for $\{\xi \mid \mathrm{ord}(\xi) > 0\}$. Prove that $k((X)) = \mathcal{M}^{-1} \cup \mathcal{U} \cup \mathcal{M}$ (disjointly), where

$$\mathcal{M}^{-1} = \{\xi \mid 1/\xi \in \mathcal{M}\}.$$

Now fix a real number, c, with $0 < c < 1$. Define for $\xi, \eta \in k((X))$,

$$d(\xi, \eta) = c^{\mathrm{ord}(\xi - \eta)},$$

then it should be clear that $k((X))$ becomes a metric space and that addition and multiplication are continuous in the metric topology. Prove that $k((X))$ is complete in this topology (i.e., Cauchy sequences converge), and that the topology is independent of which number c, $0 < c < 1$ is chosen.

(c) Suppose $u \in k[[X]]$, $u = \sum_{j=0}^{\infty} a_j X^j$, and $a_0 = 1$. Pick an integer $n \in \mathbb{Z}$ and assume $(n, \mathrm{char}(k)) = 1$. Prove: there exists $w \in k[[X]]$ such that $w^2 = u$. There is a condition on k so that $k((X))$ is locally compact. What is it? Give the proof. As an example of limiting operations, prove

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} X^j = \lim_{N \to \infty} (1 + X + \cdots + X^N).$$

(d) Given $\sum_{j=-\infty}^{\infty} a_j X^j \in k((X))$, its derivative is defined formally as

$$\sum_{j=-\infty}^{\infty} j a_j X^{j-1} \in k((X)).$$

Assume $\mathrm{ch}(k) = 0$. Check mentally that $\alpha' = 0$ $\big(\alpha \in k((X))\big) \implies \alpha \in k$. Is the map $\alpha \mapsto \alpha'$ a *continuous* linear transformation $k((X)) \to k((X))$? Set $\eta = \sum_{j=0}^{\infty} \frac{1}{j!} X^j$, so $\eta \in k((X))$. Prove that X and η are independent transcendentals over k.

(e) A topological ring is one where addition and multiplication are continuous and we have a Hausdorff topology. Topological k-algebras (k has the discrete topology) form a category in which the morphisms are *continuous* k-algebra homomorphisms. An element λ in such a ring is *topologically nilpotent* iff $\lim_{n\to\infty} \lambda^n = 0$. Let $\mathcal{N}_{\mathrm{top}}$ denote the functor which associates to each topological

k-algebra the set of its topological nilpotent elements. Prove that $\mathcal{N}_{\text{top}}$ is representable. As an application, let

$$s(X) = \sum_{j=0}^{\infty}(-1)^j \frac{X^{2j+1}}{(2j+1)!}, \quad c(X) = \sum_{j=0}^{\infty}(-1)^j \frac{X^{2j}}{(2j)!}.$$

Then $s'(X) = c(X)$ and $c'(X) = -s(X)$, so $c^2(X) + s^2(X)$ lies in k (the constants). Without computing $c^2(X) + s^2(X)$, show it is 1. (You'll need $\mathcal{N}_{\text{top}}$, so be careful.)

(f) Show that even though $k(X)$ is dense in $k((X))$, the field $k((X))$ possesses infinitely many independent transcendental elements over $k(X)$. (Suggestion: look in a number theory book under "Liouville numbers"; mimic what you find there.)

(g) Let $C_k\big(k((X))\big) = \big\{\alpha \in k((X)) \mid \alpha \text{ is algebraic over } k\big\}$. Show that $C_k\big(k((X))\big) = k$.

If $\mathrm{ch}(k) = 0$ and $\mathbb{R} \subseteq k$, write $\binom{m}{j} = \dfrac{m(m-1)\cdots(m-j+1)}{j(j-1)\cdots 3\cdot 2\cdot 1}$ for $m \in \mathbb{R}$. If $\mathbb{R} \not\subseteq k$, do this only for $m \in \mathbb{Q}$. Set

$$y_m = \sum_{j=0}^{\infty} \binom{m}{j} X^j \in k[[X]].$$

If $m = r/s$, prove that $y_m^s = (1+x)^r$.

Note that $y_m = 1 + O(X)$ and that $O(X) \in \mathcal{N}_{\text{top}}\big(k[[X]]\big)$. Let $L(1+X) = \sum_{j=0}^{\infty}(-1)^j \dfrac{X^{j+1}}{(j+1)}$, and set $f(X)^m = \eta\big(m \cdot L(f(X))\big)$, where

$$\eta(X) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j, \quad f(X) = 1 + O(X), \text{ some } O(X)$$

and $m \in \mathbb{R}$ (here, $\mathbb{R} \subseteq k$). Show that

$$(1+X)^m = y_m.$$

BIII) Say K is a field, A is a subring of K. Write $k = \operatorname{Frac} A$.

(a) If K is a finitely generated A-module, prove that $k = A$.

(b) Suppose there exists finitely many elements $\alpha_1, \ldots, \alpha_m \in K$ *algebraic over* k such that

$$K = A[\alpha_1, \ldots, \alpha_m].$$

Prove $(\exists\, b \in A)(b \neq 0)\big(\text{so that } k = A[1/b]\big)$. Prove, moreover, that b belongs to every maximal ideal of A.

BIV) Refer to AI. Look at $\mathcal{P}(X, \mathcal{A}\mathrm{b})$.

(a) We have a functor $\mathcal{P}(X, \mathcal{A}\mathrm{b}) \rightsquigarrow \mathcal{A}\mathrm{b}$ for each $U \in \mathcal{O}\mathrm{b}\, \mathcal{T}_X$, namely, $F \rightsquigarrow F(U)$. Show this functor is representable.

(b) Grothendieck realized that when computing algebraic invariants of a "space" (say homology, cohomology, homotopy, K-groups, ...) the sheaf theory one needs to use could be done far more generally and with far more richness if one abstracted the notion of "topology". Here is the generalization:

i. Replace $\mathcal{T}_X$ by *any* category $\mathcal{T}$.

To do sheaves, we need a notion of "covering":

ii. We isolate for each $U \in \mathcal{O}\mathrm{b}\,\mathcal{T}$ some families of morphisms $\{U_\alpha \to U\}_\alpha$ and call each of these a "covering" of U. So we get a whole collection of families of morphisms called $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ *and we require*

A. Any isomorphism $\{V \to U\}$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$

B. If $\{U_\alpha \to U\}_\alpha$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ and for all α, $\{V_\beta \to U_\alpha\}_\beta$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$, then $\left\{V_\beta^{(\alpha)} \to U_\alpha\right\}_{\alpha,\beta}$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ (a covering of a covering is a covering).

C. If $\{U_\alpha \to U\}_\alpha$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ and $V \to U$ is *arbitrary* then $U_\alpha \prod_U V$ exists in $\mathcal{T}$ *and*

$$\left\{U_\alpha \prod_U V \to V\right\}_\alpha$$

is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ (restriction of a covering to V is a covering of V; this allows the relative topology—it is the axiom with teeth).

Intuition: A morphism $V \to U$ in $\mathcal{T}$ is an "open subset of U". N.b. the same V and U can give more than one "open subset" (vary the morphism) so the theory is very rich. Our original example: $\mathcal{T} = \mathcal{T}_X$. The family $\{U_\alpha \to U\}_\alpha$ is in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ when and only when $\bigcup_\alpha U_\alpha = U$. Check the axioms A, B and C.

Now a presheaf is just a cofunctor $\mathcal{T} \to \mathcal{S}\mathrm{ets}$ or $\mathcal{A}\mathrm{b}$, etc. and a sheaf is a presheaf for which

$$\text{(S)} \qquad F(U) \to \prod_\gamma F\big(U_\gamma\big) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{\alpha,\beta} F\Big(U_\alpha \prod_U U_\beta\Big)$$

is exact for *every* $U \in \mathcal{T}$ and *every* $\{U_\gamma \to U\}_\gamma$ in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$. One calls the category $\mathcal{T}$ and its distinguished families $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ a *site* (topology used to be called "analysis situs")

Now given a category, say $\mathcal{T}$, assume $\mathcal{T}$ has finite fibred products. A family of morphisms $\{U_\alpha \to U\}_\alpha$ in $\mathcal{T}$ is called a family of *universal, effective epimorphisms* iff

i. $\forall Z \in \mathcal{O}\mathrm{b}\,\mathcal{T}$

$$\mathrm{Hom}(U, Z) \to \prod_\gamma \mathrm{Hom}\big(U_\gamma, Z\big) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}\big(U_\alpha \prod_U U_\beta, Z\big)$$

is exact (in $\mathcal{S}\mathrm{ets}$) AND

ii. the same for $\left\{U_\alpha \prod_U V \to V\right\}_\alpha$ *vis a vis* all Z as in i. (ii. expresses universality, i. expresses effectivity of epimorphisms).

Decree that $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ is to consist of families of universal, effective epimorphisms. Show that $\mathcal{T}$ with this $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ is a site—it is called the *canonical site on* $\mathcal{T}$, denoted $\mathcal{T}_{\mathrm{can}}$.

(c) For $\mathcal{T}_{\mathrm{can}}$, every representable cofunctor on $\mathcal{T}$ is a sheaf (give the *easy* proof). Note that if $\mathcal{T} \subseteq \widetilde{\mathcal{T}}$ where $\widetilde{\mathcal{T}}$ is a bigger category, and if $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ lies in the universal, effective epimorphisms for $\widetilde{\mathcal{T}}$, then any cofunctor on $\mathcal{T}$, *representable in* $\widetilde{\mathcal{T}}$, is a sheaf on $\mathcal{T}_{\mathrm{can}}$. For example, prove that if $\widetilde{\mathcal{T}}$ is all topological spaces and $\mathcal{T}_X$ our beginning category of AI), then $\mathcal{T}_X \subseteq \widetilde{\mathcal{T}}$ and prove: open coverings in $\mathcal{T}_X$ (as in AI) *are* universal, effective epimorphisms in $\widetilde{\mathcal{T}}$. Hence, for ANY topological space, Y, $U \rightsquigarrow \mathrm{Hom}_{\text{top.spaces}}(U, Y)$ *is a sheaf* on $\mathcal{T}_X$.

(d) Let $\mathcal{T} = \mathcal{S}\mathrm{ets}$ and let $\{U_\alpha \to U\}_\alpha$ be in $\mathcal{C}\mathrm{ov}\,\mathcal{T}$ when $\bigcup_\alpha(\text{Images of } U_\alpha) = U$. Prove that the sheaves on $\mathcal{T}$ with values in $\mathcal{S}\mathrm{ets}$ are exactly the representable cofunctors on $\mathcal{T}$.

(e) Generalize (d): G is a given group, $\mathcal{T}_G$ is the category of sets with a G-action. Make $(\mathcal{T}_G)_{\mathrm{can}}$ the canonical site on $\mathcal{T}_G$. Prove: coverings are families $\{U_\alpha \to U\}_\alpha$ so that $\bigcup_\alpha(\mathrm{Im}\ U_\alpha) = U$ (all are G-sets, morphisms are G-morphisms). Once again, prove: the sheaves on $(\mathcal{T}_G)_{\mathrm{can}}$ are exactly the representable cofunctors on $\mathcal{T}_G$. Prove further: the sheaves on $(\mathcal{T}_G)_{\mathrm{can}}$ with values in $\mathcal{A}\mathrm{b}$ form a category equivalent to the category of G-modules; namely sheaf $\rightsquigarrow$ representable cofunctor $\rightsquigarrow$ representing object, a G-module.