Spring 2023 CIS 610

Advanced Geometric Methods in Computer Science

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Homework 2

February 23; Due March 14, 2023

Problem B1 (60). (a) Consider the map, $f\colon \mathbf{GL}(n,\mathbb{R}) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of B, for any matrix B (here, I is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n,\mathbb{R})$.

(b) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n,\mathbb{R})$ is a manifold of dimension $n^2 - 1$.

(c) Let J be the $(n+1)\times(n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n,1)$ the group of real $(n+1)\times(n+1)$ matrices

$$\mathbf{SO}(n,1) = \{A \in \mathbf{GL}(n+1,\mathbb{R}) \mid A^\top J A = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n,1)$ is indeed a group with the inverse of A given by $A^{-1} = JA^\top J$ (this is the *special Lorentz group*.) Consider the function $f\colon \mathbf{GL}^+(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^\top J A - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1)\times(n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^\top J H + H^\top J A$$

for any matrix, H. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n,1)$ and that $\mathbf{SO}(n,1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

Problem B2 (30). (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and ω is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^B = \cosh \omega \, I + \frac{\sinh\,\omega}{\omega}\, B,$$

and $e^B = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^B) = 2\cosh\,\omega$.

Prove that the exponential map, $\exp\colon \mathfrak{sl}(2, \mathbb{C}) \to \mathbf{SL}(2, \mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2, \mathbb{C})$.

Problem B3 (30). Consider the parametric surface given by

$$\begin{aligned} x(u, v) &= \frac{8uv}{(u^2 + v^2 + 1)^2}, \\ y(u, v) &= \frac{4v(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\ z(u, v) &= \frac{4(u^2 - v^2)}{(u^2 + v^2 + 1)^2}. \end{aligned}$$

The trace of this surface is called a *crosscap*. In order to plot this surface, make the change of variables

$$\begin{aligned} u &= \rho \cos\theta \\ v &= \rho \sin\theta. \end{aligned}$$

Prove that we obtain the parametric definition

$$\begin{aligned} x &= \frac{4\rho^2}{(\rho^2 + 1)^2} \sin 2\theta, \\ y &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \sin\theta, \\ z &= \frac{4\rho^2}{(\rho^2 + 1)^2} \cos 2\theta. \end{aligned}$$

Show that the entire trace of the surface is obtained for $\rho \in [0, 1]$ and $\theta \in [-\pi, \pi]$.

Hint. What happens if you change ρ to $1/\rho$?

Plot the trace of the surface using the above parametrization. Show that there is a line of self-intersection along the portion of the z-axis corresponding to $0 \leq z \leq 1$. What can you say about the point corresponding to $\rho = 1$ and $\theta = 0$?

Plot the portion of the surface for $\rho \in [0, 1]$ and $\theta \in [0, \pi]$.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$\begin{aligned} x &= \frac{16uv^2(1-u^2)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

Problem B4 (30). Consider the parametric surface given by

$$\begin{aligned} x(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

The trace of this surface is called the *Steiner Roman surface*. In order to plot this surface, make the change of variables

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned}$$

Prove that we obtain the parametric definition

$$\begin{aligned} x &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\sin\theta, \\ y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\cos\theta, \\ z &= \frac{4\rho^2}{(\rho^2+1)^2}\,\cos 2\theta. \end{aligned}$$

Show that the entire trace of the surface is obtained for $\rho \in [0, 1]$ and $\theta \in [-\pi, \pi]$. Plot the trace of the surface using the above parametrization.

Plot the portion of the surface for $\rho \in [0, 1]$ and $\theta \in [0, \pi]$.

Prove that this surface has five singular points.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$x = \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2},$$
$$y = \frac{4v(1-u^4)(v^2-1)}{(u^2+1)^2(v^2+1)^2},$$
$$z = \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}.$$

Problem B5 (160). Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

(a) Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\begin{aligned}
\psi_1(u,v) &= \left(\frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1}\right),\\
\psi_2(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{u^2-1}{u^2+v^2+1}\right),\\
\psi_3(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1-u^2}{u^2+v^2+1}\right).
\end{aligned}$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}\right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3 \colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u,v) \mapsto \left(\frac{1}{\sqrt{u^2+v^2+1}}, \frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}\right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian has rank 2).

(c) Prove that if $\psi_1(u,v) = (x,y,z,t)$, then

$$y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that if $\psi_1(u,v) = (x,y,z,t)$, then u and v satisfy the equations

$$\begin{aligned} (y^2 + z^2)u^2 - zu + z^2 &= 0 \\ (y^2 + z^2)v^2 - yv + y^2 &= 0. \end{aligned}$$

Prove that if $y^2 + z^2 \neq 0$, then

$$u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$

else

$$u = \frac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1,$$

and there are similar formulae for v. Prove that the expression giving u in terms of y and z is continuous everywhere in $\{(y,z) \mid y^2 + z^2 \leq \frac{1}{4}\}$ and similarly for the expression giving v in terms of y and z. Conclude that $\psi_1 \colon \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)$ is a homeomorphism onto its image. Therefore, $U_1 = \psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_2(u,v) = (x,y,z,t)$, then

$$x^2 + y^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + y^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that if $\psi_2(u,v) = (x,y,z,t)$, then u and v satisfy the equations

$$\begin{aligned} (x^2 + y^2)u^2 - xu + x^2 &= 0 \\ (x^2 + y^2)v^2 - yv + y^2 &= 0. \end{aligned}$$

Conclude that $\psi_2 \colon \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_2 = \psi_2(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_3(u,v) = (x,y,z,t)$, then

$$x^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that if $\psi_3(u,v) = (x,y,z,t)$, then u and v satisfy the equations

$$\begin{aligned} (x^2 + z^2)u^2 - xu + x^2 &= 0 \\ (x^2 + z^2)v^2 - zv + z^2 &= 0. \end{aligned}$$

Conclude that $\psi_3\colon \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_3 = \psi_3(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that the union of the U_i's covers $\mathcal{H}(S^2)$. Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a smooth manifold in $\mathbb{R}^4$.

(d) Plot the surfaces obtained by dropping the fourth coordinate and the third coordinates, respectively (with $u, v \in [-1, 1]$).

(e) Prove that if $(x, y, z, t) \in \mathcal{H}(S^2)$, then

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= xyz \\ x(z^2 - y^2) &= yzt. \end{aligned}$$

Prove that the zero locus of these equations strictly contains $\mathcal{H}(S^2)$. This is a "famous mistake" of Hilbert and Cohn-Vossen in *Geometry and the Immagination*!

Finding a set of equations defining exactly $\mathcal{H}(S^2)$ appears to be an open problem.

Problem B6 (60 pts). We can let the group $\mathbf{SO}(3)$ act on itself by conjugation, so that

$$R \cdot S = RSR^{-1} = RSR^{\top}.$$

The orbits of this action are the *conjugacy classes* of $\mathbf{SO}(3)$.

(1) Prove that the conjugacy classes of $\mathbf{SO}(3)$ are in bijection with the following sets:

1. $\mathcal{C}_0 = \{(0, 0, 0)\}$, the sphere of radius 0.
2. $\mathcal{C}_\theta$, with $0 < \theta < \pi$ and
$$\mathcal{C}_\theta = \{u \in \mathbb{R}^3 \mid \|u\| = \theta\},$$
the sphere of radius θ.
3. $\mathcal{C}_\pi = \mathbb{RP}^2$, viewed as the quotient of the sphere of radius π by the equivalence relation of being antipodal.

(2) Give $\mathrm{M}_3(\mathbb{R})$ the Euclidean structure where

$$\langle A, B\rangle = \frac{1}{2}\mathrm{tr}(A^{\top}B).$$

Consider the following three curves in $\mathbf{SO}(3)$:

$$c(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $0 \leq t \leq 2\pi$,

$$\alpha(\theta) = \begin{pmatrix} -\cos 2\theta & 0 & \sin 2\theta \\ 0 & -1 & 0 \\ \sin 2\theta & 0 & \cos 2\theta \end{pmatrix},$$

for $-\pi/2 \leq \theta \leq \pi/2$, and

$$\beta(\theta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

for $-\pi/2 \leq \theta \leq \pi/2$.

Check that $c(t)$ is a rotation of angle t and axis $(0, 0, 1)$, that $\alpha(\theta)$ is a rotation of angle π whose axis is in the (x, z)-plane, and that $\beta(\theta)$ is a rotation of angle π whose axis is in the (y, z)-plane. Show that a log of $\alpha(\theta)$ is

$$B_\alpha = \pi \begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix},$$

and that a log of $\beta(\theta)$ is

$$B_\beta = \pi \begin{pmatrix} 0 & -\cos\theta & \sin\theta \\ \cos\theta & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix}.$$

(3) The curve $c(t)$ is a closed curve starting and ending at I that intersects $\mathcal{C}_\pi$ for $t = \pi$, and α, β are contained in $\mathcal{C}_\pi$ and coincide with $c(\pi)$ for $\theta = 0$. Compute the derivative $c'(\pi)$ of $c(t)$ at $t = \pi$, and the derivatives $\alpha'(0)$ and $\beta'(0)$, and prove that they are pairwise orthogonal (under the inner product $\langle -, - \rangle$).

Conclude that $c(t)$ intersects $\mathcal{C}_\pi$ transversally in $\mathbf{SO}(3)$, which means that

$$T_{c(\pi)}\, c + T_{c(\pi)}\, \mathcal{C}_\pi = T_{c(\pi)}\, \mathbf{SO}(3).$$

This fact can be used to prove that all closed curves smoothly homotopic to $c(t)$ must intersect $\mathcal{C}_\pi$ transversally, and consequently $c(t)$ is not (smoothly) homotopic to a point. This implies that $\mathbf{SO}(3)$ is not simply connected, but this will have to wait for another homework!

TOTAL: 370 **points.**