Chapter 6

The Quaternions and the Spaces S^3, $\mathbf{SU}(2)$, $\mathbf{SO}(3)$, and $\mathbb{RP}^3$

6.1 The Algebra $\mathbb{H}$ of Quaternions

In this chapter, we discuss the representation of rotations of $\mathbb{R}^3$ and $\mathbb{R}^4$ in terms of quaternions.

Such a representation is not only concise and elegant, it also yields a very efficient way of handling composition of rotations.

It also tends to be numerically more stable than the representation in terms of orthogonal matrices.

The group of rotations $\mathbf{SO}(2)$ is isomorphic to the group $\mathbf{U}(1)$ of complex numbers $e^{i\theta} = \cos\theta + i\sin\theta$ of unit length. This follows imediately from the fact that the map

$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a group isomorphism.

Geometrically, observe that $\mathbf{U}(1)$ is the unit circle S^1.

We can identify the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, letting $z = x + iy \in \mathbb{C}$ represent $(x, y) \in \mathbb{R}^2$.

Then, every plane rotation ρ_θ by an angle θ is represented by multiplication by the complex number $e^{i\theta} \in \mathbf{U}(1)$, in the sense that for all $z, z' \in \mathbb{C}$,

$$z' = \rho_\theta(z) \quad \text{iff} \quad z' = e^{i\theta} z.$$

In some sense, the quaternions generalize the complex numbers in such a way that rotations of $\mathbb{R}^3$ are represented by multiplication by quaternions of unit length. This is basically true with some twists.

For instance, quaternion multiplication is not commutative, and a rotation in $\mathbf{SO}(3)$ requires conjugation with a (unit) quaternion for its representation.

Instead of the unit circle S^1, we need to consider the sphere S^3 in $\mathbb{R}^4$, and $\mathbf{U}(1)$ is replaced by $\mathbf{SU}(2)$.

Recall that the 3-sphere S^3 is the set of points $(x, y, z, t) \in \mathbb{R}^4$ such that

$$x^2 + y^2 + z^2 + t^2 = 1,$$

and that the real projective space $\mathbb{RP}^3$ is the quotient of S^3 modulo the equivalence relation that identifies antipodal points (where (x, y, z, t) and $(-x, -y, -z, -t)$ are antipodal points).

The group $\mathbf{SO}(3)$ of rotations of $\mathbb{R}^3$ is intimately related to the 3-sphere S^3 and to the real projective space $\mathbb{RP}^3$.

The key to this relationship is the fact that rotations can be represented by quaternions, discovered by Hamilton in 1843.

Historically, the quaternions were the first instance of a noncommutative field. As we shall see, quaternions represent rotations in $\mathbb{R}^3$ very concisely.

It will be convenient to define the quaternions as certain 2×2 complex matrices.

We write a complex number z as $z = a + ib$, where $a, b \in \mathbb{R}$, and the *conjugate* $\overline{z}$ of z is $\overline{z} = a - ib$.

Let $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the following matrices:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Consider the set $\mathbb{H}$ of all matrices of the form

$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $(a, b, c, d) \in \mathbb{R}^4$. Every matrix in $\mathbb{H}$ is of the form

$$A = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix},$$

where $x = a + ib$ and $y = c + id$. The matrices in $\mathbb{H}$ are called *quaternions*.

The null quaternion is denoted as 0 (or $\mathbf{0}$, if confusions arise).

Quaternions of the form $b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ are called *pure quaternions*. The set of pure quaternions is denoted as $\mathbb{H}_p$.

Note that the rows (and columns) of such matrices are vectors in $\mathbb{C}^2$ that are orthogonal with respect to the Hermitian inner product of $\mathbb{C}^2$ given by

$$(x_1, y_1).(x_2, y_2) = x_1\overline{x_2} + y_1\overline{y_2}.$$

Furthermore, their norm is

$$\sqrt{x\overline{x} + y\overline{y}} = \sqrt{a^2 + b^2 + c^2 + d^2},$$

and the determinant of A is $a^2 + b^2 + c^2 + d^2$.

It is easily seen that the following famous identities (discovered by Hamilton) hold:

$$\begin{aligned}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} &= -\mathbf{1}\\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k}\\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i}\\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}.\end{aligned}$$

Using these identities, it can be verified that $\mathbb{H}$ is a ring (with multiplicative identity $\mathbf{1}$) and a real vector space of dimension 4 with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$.

In fact, $\mathbb{H}$ is an associative algebra. For details, see Berger [**?**], Veblen and Young [**?**], Dieudonné [**?**], Bertin [**?**].

The quaternions $\mathbb{H}$ are often defined as the real algebra generated by the four elements $\mathbf{1}$, $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and satisfying the identities just stated above.

The problem with such a definition is that it is not obvious that the algebraic structure $\mathbb{H}$ actually exists.

A rigorous justification requires the notions of freely generated algebra and of quotient of an algebra by an ideal.

Our definition in terms of matrices makes the existence of $\mathbb{H}$ trivial (but requires showing that the identities hold, which is an easy matter).

Given any two quaternions $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = a'\mathbf{1} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, it can be verified that

$$\begin{aligned} XY = {} & (aa' - bb' - cc' - dd')\mathbf{1} + (ab' + ba' + cd' - dc')\mathbf{i} \\ & + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k}. \end{aligned}$$

It is worth noting that these formulae were discovered independently by Olinde Rodrigues in 1840, a few years before Hamilton (Veblen and Young [**?**]).

However, Rodrigues was working with a different formalism, homogeneous transformations, and he did not discover the quaternions.

The map from $\mathbb{R}$ to $\mathbb{H}$ defined such that $a \mapsto a\mathbf{1}$ is an injection which allows us to view $\mathbb{R}$ as a subring $\mathbb{R}\mathbf{1}$ (in fact, a field) of $\mathbb{H}$.

Similarly, the map from $\mathbb{R}^3$ to $\mathbb{H}$ defined such that $(b, c, d) \mapsto b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is an injection which allows us to view $\mathbb{R}^3$ as a subspace of $\mathbb{H}$, in fact, the hyperplane $\mathbb{H}_p$.

Given a quaternion $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we define its *conjugate* $\overline{X}$ as

$$\overline{X} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

It is easily verified that

$$X\overline{X} = (a^2 + b^2 + c^2 + d^2)\mathbf{1}.$$

The quantity $a^2 + b^2 + c^2 + d^2$, also denoted as $N(X)$, is called the *reduced norm* of X.

Clearly, X is nonnull iff $N(X) \neq 0$, in which case $\overline{X}/N(X)$ is the multiplicative inverse of X.

Thus, $\mathbb{H}$ is a noncommutative field.

Since $X + \overline{X} = 2a\mathbf{1}$, we also call $2a$ the *reduced trace* of X, and we denote it as $Tr(X)$.

A quaternion X is a pure quaternion iff $\overline{X} = -X$ iff $Tr(X) = 0$. The following identities can be shown (see Berger [**?**], Dieudonné [**?**], Bertin [**?**]):

$$
\begin{aligned}
\overline{XY} &= \overline{Y}\,\overline{X}, \\
Tr(XY) &= Tr(YX), \\
N(XY) &= N(X)N(Y), \\
Tr(ZXZ^{-1}) &= Tr(X),
\end{aligned}
$$

whenever $Z \neq 0$.

If $X = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, are pure quaternions, identifying X and Y with the corresponding vectors in $\mathbb{R}^3$, the inner product $X \cdot Y$ and the cross-product $X \times Y$ make sense, and letting $[0, X \times Y]$ denote the quaternion whose first component is 0 and whose last three components are those of $X \times Y$, we have the remarkable identity

$$XY = -(X \cdot Y)\mathbf{1} + [0, X \times Y].$$

More generally, given a quaternion $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we can write it as

$$X = [a, (b, c, d)],$$

where a is called the *scalar part* of X and (b, c, d) the *pure part* of X.

Then, if $X = [a, U]$ and $Y = [a', U']$, it is easily seen that the quaternion product XY can be expressed as

$$XY = [aa' - U \cdot U',\, aU' + a'U + U \times U'].$$

The above formula for quaternion multiplication allows us to show the following fact.

Let $Z \in \mathbb{H}$, and assume that $ZX = XZ$ for all $X \in \mathbb{H}$. Then, the pure part of Z is null, i.e., $Z = a\mathbf{1}$ for some $a \in \mathbb{R}$.

Remark: It is easy to check that for arbitrary quaternions $X = [a, U]$ and $Y = [a', U']$,

$$XY - YX = [0, 2(U \times U')],$$

and that for pure quaternion $X, Y \in \mathbb{H}_p$,

$$2(X \cdot Y)\mathbf{1} = -(XY + YX).$$

Since quaternion multiplication is bilinear, for a given X, the map $Y \mapsto XY$ is linear, and similarly for a given Y, the map $X \mapsto XY$ is linear. If the matrix of the first map is L_X and the matrix of the second map is R_Y, then

$$XY = L_X Y = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}$$

and

$$XY = R_Y X = \begin{pmatrix} a' & -b' & -c' & -d' \\ b' & a' & d' & -c' \\ c' & -d' & a' & b' \\ d' & c' & -b' & a' \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Observe that the columns (and the rows) of the above matrices are orthogonal.

Thus, when X and Y are unit quaternions, both L_X and R_Y are orthogonal matrices. Furthermore, it is obvious that $L_{\overline{X}} = L_X^\top$, the transpose of L_X, and similarly $R_{\overline{Y}} = R_Y^\top$.

It is easily shown that

$$\det(L_X) = (a^2 + b^2 + c^2 + d^2)^2.$$

This shows that when X is a unit quaternion, L_X is a rotation matrix, and similarly when Y is a unit quaternion, R_Y is a rotation matrix (see Veblen and Young [**?**]).

Define the map $\varphi\colon \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ as follows:

$$\varphi(X, Y) = \frac{1}{2} Tr(X\,\overline{Y}) = aa' + bb' + cc' + dd'.$$

It is easily verified that φ is bilinear, symmetric, and definite positive. Thus, the quaternions form a Euclidean space under the inner product defined by φ (see Berger [**?**], Dieudonné [**?**], Bertin [**?**]).

It is immediate that under this inner product, the norm of a quaternion X is just $\sqrt{N(X)}$.

It is also immediate that the set of pure quaternions is orthogonal to the space of "real quaternions" $\mathbb{R}\mathbf{1}$.

As a Euclidean space, $\mathbb{H}$ is isomorphic to $\mathbb{E}^4$.

It is also immediate that the subspace $\mathbb{H}_p$ of pure quaternions is orthogonal to the space of "real quaternions" $\mathbb{R}\mathbf{1}$.

The subspace $\mathbb{H}_p$ of pure quaternions inherits a Euclidean structure, and this subspace is isomorphic to the Euclidean space $\mathbb{E}^3$.

Since $\mathbb{H}$ and $\mathbb{E}^4$ are isomorphic Euclidean spaces, their groups of rotations $\mathbf{SO}(\mathbb{H})$ and $\mathbf{SO}(4)$ are isomorphic, and we will identify them.

Similarly, we will identify $\mathbf{SO}(\mathbb{H}_p)$ and $\mathbf{SO}(3)$.

6.2 Quaternions and Rotations in $\mathbf{SO}(3)$

We just observed that for any nonnull quaternion X, both maps $Y \mapsto XY$ and $Y \mapsto YX$ (where $Y \in \mathbb{H}$) are linear maps, and that when $N(X) = 1$, these linear maps are in $\mathbf{SO}(4)$.

This suggests looking at maps $\rho_{Y,Z}\colon \mathbb{H} \to \mathbb{H}$ of the form $X \mapsto YXZ$, where $Y, Z \in \mathbb{H}$ are any two fixed nonnull quaternions such that $N(Y)N(Z) = 1$.

In view of the identity $N(UV) = N(U)N(V)$ for all $U, V \in \mathbb{H}$, we see that $\rho_{Y,Z}$ is an isometry.

In fact, since

$$\rho_{Y,Z} = \rho_{Y,\mathbf{1}} \circ \rho_{\mathbf{1},Z},$$

$\rho_{Y,Z}$ itself is a rotation, i.e. $\rho_{Y,Z} \in \mathbf{SO}(4)$.

We will prove that every rotation in $\mathbf{SO}(4)$ arises in this fashion.

Also, observe that when $Z = Y^{-1}$, the map $\rho_{Y,Y^{-1}}$, denoted more simply as ρ_Y, is the identity on $\mathbf{1}\mathbb{R}$, and maps $\mathbb{H}_p$ into itself.

Thus, $\rho_Z \in \mathbf{SO}(3)$, i.e., ρ_Z is a rotation of $\mathbb{E}^3$.

We will prove that every rotation in $\mathbf{SO}(3)$ arises in this fashion.

The quaternions of norm 1, also called *unit quaternions*, are in bijection with points of the real 3-sphere S^3.

It is easy to verify that the unit quaternions form a subgroup of the multiplicative group $\mathbb{H}^*$ of nonnull quaternions. In terms of complex matrices, the unit quaternions correspond to the group of unitary complex 2×2 matrices of determinant 1 (i.e., $x\overline{x} + y\overline{y} = 1$)

$$A = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix},$$

with respect to the Hermitian inner product in $\mathbb{C}^2$.

This group is denoted as $\mathbf{SU}(2)$.

The obvious bijection between $\mathbf{SU}(2)$ and S^3 is in fact a homeomorphism, and it can be used to transfer the group structure on $\mathbf{SU}(2)$ to S^3, which becomes a topological group isomorphic to the topological group $\mathbf{SU}(2)$ of unit quaternions.

It should also be noted that the fact that the shere S^3 has a group structure is quite exceptional.

As a matter of fact, the only spheres for which a continuous group structure is definable are S^1 and S^3.

One of the most important properties of the quaternions is that they can be used to represent rotations of $\mathbb{R}^3$, as stated in the following lemma.

Lemma 6.2.1 *For every quaternion $Z \neq 0$, the map*

$$\rho_Z \colon X \mapsto ZXZ^{-1}$$

(where $X \in \mathbb{H}$) is a rotation in $\mathbf{SO}(\mathbb{H}) = \mathbf{SO}(4)$ whose restriction to the space $\mathbb{H}_p$ of pure quaternions is a rotation in $\mathbf{SO}(\mathbb{H}_p) = \mathbf{SO}(3)$. Conversely, every rotation in $\mathbf{SO}(3)$ is of the form

$$\rho_Z \colon X \mapsto ZXZ^{-1},$$

for some quaternion $Z \neq 0$, and for all $X \in \mathbb{H}_p$. Furthermore, if two nonnull quaternions Z and Z' represent the same rotation, then $Z' = \lambda Z$ for some $\lambda \neq 0$ in $\mathbb{R}$.

As a corollary of

$$\rho_{YX} = \rho_Y \circ \rho_X,$$

it is easy to show that the map

$$\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$$

defined such that $\rho(Z) = \rho_Z$ is a surjective and continuous homomorphism whose kernel is $\{\mathbf{1}, -\mathbf{1}\}$.

Since $\mathbf{SU}(2)$ and S^3 are homeomorphic as topological spaces, this shows that $\mathbf{SO}(3)$ is homeomorphic to the quotient of the sphere S^3 modulo the antipodal map.

But the real projective space $\mathbb{RP}^3$ is defined precisely this way in terms of the antipodal map $\pi\colon S^3 \to \mathbb{RP}^3$, and thus $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are homeomorphic.

This homeomorphism can then be used to transfer the group structure on $\mathbf{SO}(3)$ to $\mathbb{RP}^3$ which becomes a topological group.

Moreover, it can be shown that $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are diffeomorphic manifolds (see Marsden and Ratiu [**?**]).

Thus, $\mathbf{SO}(3)$ and $\mathbb{RP}^3$ are at the same time, groups, topological spaces, and manifolds, and in fact they are Lie groups (see Marsden and Ratiu [**?**] or Bryant [**?**]).

The axis and the angle of a rotation can also be extracted from a quaternion representing that rotation.

Lemma 6.2.2 *For every quaternion $Z = a\mathbf{1}+t$ where t is a nonnull pure quaternion, the axis of the rotation ρ_Z associated with Z is determined by the vector in $\mathbb{R}^3$ corresponding to t, and the angle of rotation θ is equal to π when $a = 0$, or when $a \neq 0$, given a suitable orientation of the plane orthogonal to the axis of rotation, by*

$$\tan\frac{\theta}{2} = \frac{\sqrt{N(t)}}{|a|},$$

with $0 < \theta \leq \pi$.

We can write the unit quaternion Z as

$$Z = \left[\cos\frac{\theta}{2},\, \sin\frac{\theta}{2}\, V\right],$$

where V is the unit vector $\frac{t}{\sqrt{N(t)}}$ (with $-\pi \leq \theta \leq \pi$).

Also note that $VV = -\mathbf{1}$, and thus, formally, every unit quaternion looks like a complex number $\cos\varphi + i\sin\varphi$, except that i is replaced by a unit vector, and multiplication is quaternion multiplication.

In order to explain the homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ more concretely, we now derive the formula for the rotation matrix of a rotation ρ whose axis D is determined by the nonnull vector w and whose angle of rotation is θ.

For simplicity, we may assume that w is a unit vector.

Letting $W = (b, c, d)$ be the column vector representing w and H be the plane orthogonal to w, recall that the matrices representing the projections p_D and p_H are

$$WW^\top \quad \text{and} \quad I - WW^\top.$$

Given any vector $u \in \mathbb{R}^3$, the vector $\rho(u)$ can be expressed in terms of the vectors $p_D(u)$, $p_H(u)$, and $w \times p_H(u)$, as

$$\rho(u) = p_D(u) + \cos\theta\, p_H(u) + \sin\theta\, w \times p_H(u).$$

However, it is obvious that

$$w \times p_H(u) = w \times u,$$

so that

$$\rho(u) = p_D(u) + \cos\theta\, p_H(u) + \sin\theta\, w \times u,$$

and we know from Section 5.9 that the cross-product $w \times u$ can be expressed in terms of the multiplication on the left by the matrix

$$A = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

Then, letting

$$B = WW^\top = \begin{pmatrix} b^2 & bc & bd \\ bc & c^2 & cd \\ bd & cd & d^2 \end{pmatrix},$$

the matrix R representing the rotation ρ is

$$\begin{aligned} R &= WW^\top + \cos\theta(I - WW^\top) + \sin\theta A, \\ &= \cos\theta\, I + \sin\theta A + (1-\cos\theta)WW^\top, \\ &= \cos\theta\, I + \sin\theta A + (1-\cos\theta)B. \end{aligned}$$

Thus,

$$R = \cos\theta\, I + \sin\theta A + (1-\cos\theta)B.$$

with

$$A = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

It is immediately verified that

$$A^2 = B - I,$$

and thus, R is also given by

$$R = I + \sin\theta A + (1 - \cos\theta)A^2,$$

with

$$A = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

Then, the nonnull unit quaternion

$$Z = \left[\cos\frac{\theta}{2},\, \sin\frac{\theta}{2}\, V\right],$$

where $V = (b, c, d)$ is a unit vector, corresponds to the rotation ρ_Z of matrix

$$R = I + \sin\theta A + (1 - \cos\theta)A^2.$$

with

$$A = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

Remark: A related formula known as Rodrigues' formula (1840) gives an expression for a rotation matrix in terms of the exponential of a matrix (the exponential map).

Indeed, given $(b, c, d) \in \mathbb{R}^3$, letting $\theta = \sqrt{b^2 + c^2 + d^2}$, we have

$$e^A = \cos\theta\, I + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} B,$$

with A and B as above, but (b, c, d) not necessarily a unit vector. We will study exponential maps later on.

Using the matrices L_X and R_Y introduced earlier, since $XY = L_X Y = R_Y X$, from $Y = ZXZ^{-1} = ZX\overline{Z}/N(Z)$, we get

$$Y = \frac{1}{N(Z)} L_Z R_{\overline{Z}} X.$$

Thus, if we want to see the effect of the rotation specified by the quaternion Z in terms of matrices, we simply have to compute the matrix

$$\frac{1}{N(Z)} L_Z R_{\overline{Z}}$$
$$= \frac{1}{N(Z)} \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

which yields

$$\frac{1}{N(Z)}\begin{pmatrix} N(Z) & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ 0 & -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}$$

where $N(Z) = a^2 + b^2 + c^2 + d^2$.

But since every pure quaternion X is a vector whose first component is 0, we see that the rotation matrix $R(Z)$ associated with the quaternion Z is

$$R(Z) = \frac{1}{N(Z)}\begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}$$

This expression for a rotation matrix is due to Euler (see Veblen and Young [**?**]).

It is remarkable that this matrix only contains quadratic polynomials in a, b, c, d. This makes it possible to compute easily a quaternion from a rotation matrix.

From a computational point of view, it is worth noting that computing the composition of two rotations ρ_Y and ρ_Z specified by two quaternions Y, Z using quaternion multiplication (i.e. $\rho_Y \circ \rho_Z = \rho_{YZ}$) is cheaper than using rotation matrices and matrix multiplication.

On the other hand, computing the image of a point X under a rotation ρ_Z is more expensive in terms of quaternions (it requires computing ZXZ^{-1}) than it is in terms of rotation matrices (where only AX needs to be computed, where A is a rotation matrix).

Thus, if many points need to be rotated and the rotation is specified by a quaternion, it is advantageous to precompute the Euler matrix.

6.3 Quaternions and Rotations in $\mathbf{SO}(4)$

For every nonnull quaternion Z, the map $X \mapsto ZXZ^{-1}$ (where X is a pure quaternion) defines a rotation of $\mathbb{H}_p$, and conversely every rotation of $\mathbb{H}_p$ is of the above form.

What happens if we consider a map of the form

$$X \mapsto YXZ,$$

where $X \in \mathbb{H}$, and $N(Y)N(Z) = 1$?

Remarkably, it turns out that we get all the rotations of $\mathbb{H}$.

Lemma 6.3.1 *For every pair (Y, Z) of quaternions such that $N(Y)N(Z) = 1$, the map*

$$\rho_{Y,Z}: X \mapsto YXZ$$

(where $X \in \mathbb{H}$) is a rotation in $\mathbf{SO}(\mathbb{H}) = \mathbf{SO}(4)$. Conversely, every rotation in $\mathbf{SO}(4)$ is of the form

$$\rho_{Y,Z}: X \mapsto YXZ,$$

for some quaternions Y, Z, such that $N(Y)N(Z) = 1$. Furthermore, if two nonnull pairs of quaternions (Y, Z) and (Y', Z') represent the same rotation, then $Y' = \lambda Y$ and $Z' = \lambda^{-1}Z$, for some $\lambda \neq 0$ in $\mathbb{R}$.

It is easily seen that

$$\rho_{(Y'Y, ZZ')} = \rho_{Y',Z'} \circ \rho_{Y,Z},$$

and as a corollary, it is it easy to show that the map

$$\eta: S^3 \times S^3 \to \mathbf{SO}(4)$$

defined such that $\eta(Y, Z) = \rho_{Y,\overline{Z}}$ is a surjective homomorphism whose kernel is $\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$.

We conclude this Section with a mention of the exponential map, since it has applications to quaternion interpolation, which, in turn, has applications to motion interpolation.

Observe that the quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ can also be written as

$$
\begin{aligned}
\mathbf{i} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\mathbf{j} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\mathbf{k} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\end{aligned}
$$

so that, if we define the matrices $\sigma_1, \sigma_2, \sigma_3$ such that

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

we can write

$$
Z = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a\mathbf{1} + i(d\sigma_1 + c\sigma_2 + b\sigma_3).
$$

The matrices $\sigma_1, \sigma_2, \sigma_3$ are called the *Pauli spin matrices*.

Note that their traces are null and that they are Hermitian (recall that a complex matrix is Hermitian iff it is equal to the transpose of its conjugate, i.e., $A^* = A$).

The somewhat unfortunate order reversal of b, c, d has to do with the traditional convention for listing the Pauli matrices.

If we let $e_0 = a$, $e_1 = d$, $e_2 = c$ and $e_3 = b$, then Z can be written as

$$Z = e_0\mathbf{1} + i(e_1\sigma_1 + e_2\sigma_2 + e_3\sigma_3),$$

and e_0, e_1, e_2, e_3 are called the *Euler parameters* of the rotation specified by Z.

If $N(Z) = 1$, then we can also write

$$Z = \cos\frac{\theta}{2}\,\mathbf{1} + i\sin\frac{\theta}{2}\,(\beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1),$$

where

$$(\beta, \gamma, \delta) = \frac{1}{\sin\frac{\theta}{2}}\,(b, c, d).$$

Letting $A = \beta\sigma_3 + \gamma\sigma_2 + \delta\sigma_1$, it can be shown that

$$e^{i\theta A} = \cos\theta\,\mathbf{1} + i\sin\theta\,A,$$

where the exponential is the usual exponential of matrices, i.e., for a square $n \times n$ matrix M,

$$\exp(M) = I_n + \sum_{k\geq 1}\frac{M^k}{k!}.$$

Note that since A is Hermitian of null trace, iA is skew Hermitian of null trace.

The above formula turns out to define the exponential map from the Lie Algebra of $\mathbf{SU}(2)$ to $\mathbf{SU}(2)$. The Lie algebra of $\mathbf{SU}(2)$ is a real vector space having $i\sigma_1$, $i\sigma_2$, and $i\sigma_3$, as a basis.

Now, the vector space $\mathbb{R}^3$ is a Lie algebra if we define the Lie bracket on $\mathbb{R}^3$ as the usual cross-product $u \times v$ of vectors.

Then, the Lie algebra of $\mathbf{SU}(2)$ is isomorphic to $(\mathbb{R}^3, \times)$, and the exponential map can be viewed as a map

$$\exp\colon (\mathbb{R}^3, \times) \to \mathbf{SU}(2)$$

given by the formula

$$\exp(\theta v) = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, v\right],$$

for every vector θv, where v is a unit vector in $\mathbb{R}^3$, and $\theta \in \mathbb{R}$.

6.4 Applications of Euclidean Geometry to Motion Interpolation

The exponential map can be used for quaternion interpolation.

Given two unit quaternions X, Y, suppose we want to find a quaternion Z "interpolating" between X and Y.

We have to clarify what this means.

Since $\mathbf{SU}(2)$ is topologically the same as the sphere S^3, we define an *interpolant* of X and Y as a quaternion Z on the great circle (on the sphere S^3) determined by the intersection of S^3 with the (2-)plane defined by the two points X and Y (viewed as points on S^3) and the orgin $(0, 0, 0, 0)$.

Then, the points (quaternions) on this great circle can be defined by first rotating X and Y so that X goes to $\mathbf{1}$ and Y goes to $X^{-1}Y$, by multiplying (on the left) by X^{-1}.

Letting

$$X^{-1}Y = [\cos\Omega,\ \sin\Omega\, w]\,,$$

where $-\pi < \Omega \leq \pi$, the points on the great circle from $\mathbf{1}$ to $X^{-1}Y$ are given by the quaternions

$$(X^{-1}Y)^{\lambda} = [\cos\lambda\Omega,\ \sin\lambda\Omega\, w]\,,$$

where $\lambda \in \mathbb{R}$.

This is because $X^{-1}Y = \exp(2\Omega w)$, and since an interpolant between $(0,0,0)$ and $2\Omega w$ is $2\lambda\Omega w$ in the Lie algebra of $\mathbf{SU}(2)$, the corresponding quaternion is indeed

$$\exp(2\lambda\Omega) = [\cos\lambda\Omega,\ \sin\lambda\Omega\, w]\,.$$

We can't justify all this here, but it is indeed correct.

If $\Omega \neq \pi$, then the shortest arc between X and Y is unique, and it corresponds to those λ such that $0 \leq \lambda \leq 1$ (it is a geodesic arc).

However, if $\Omega = \pi$, then X and Y are antipodal, and there are infinitely many half circles from X to Y. In this case, w can be chosen arbitrarily.

Finally, having the arc of great circle between $\mathbf{1}$ and $X^{-1}Y$ (assuming $\Omega \neq \pi$), we get the arc of interpolants $Z(\lambda)$ between X and Y by performing the inverse rotation from $\mathbf{1}$ to X and from $X^{-1}Y$ to Y, i.e., by multiplying (on the left) by X, and we get

$$Z(\lambda) = X(X^{-1}Y)^{\lambda}.$$

It is remarkable that a closed-form formula for $Z(\lambda)$ can be given, as shown by Shoemake [**?**, **?**].

If $X = [\cos\theta, \sin\theta\, u]$, and $Y = [\cos\varphi, \sin\varphi\, v]$ (where u and v are unit vectors in $\mathbb{R}^3$), letting

$$\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi\,(u\cdot v)$$

be the inner product of X and Y viewed as vectors in $\mathbb{R}^4$, it is a bit laborious to show that

$$Z(\lambda) = \frac{\sin(1-\lambda)\Omega}{\sin\Omega}\,X + \frac{\sin\lambda\Omega}{\sin\Omega}\,Y.$$