Algebraic Geometry
Class Notes For Math 622
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Chapter 1

Introduction and Motivating Examples

1.1 Affine Subvarieties

Given a field k, we denote the n-dimensional affine space over k as $\mathbb{A}^n$. Picking coordinates $z_1, \ldots, z_n$, for any nonnull polynomial $f \in k[z_1, \ldots, z_n]$, the *zero locus* of f is the set

$$X_f = \{(x_1, \ldots, x_n) \in k^n \mid f(x_1, \ldots, x_n) = 0\}.$$

This is called an *affine hypersurface*.

More generally, given a set of l polynomial $\{f_1, \ldots, f_l\}$ with $f_i \in k[z_1, \ldots, z_n]$, we define $X_{(f_1,\ldots,f_l)}$ as

$$X_{(f_1,\ldots,f_l)} = \{(x_1, \ldots, x_n) \in k^n \mid f_i(x_1, \ldots, x_n) = 0,\ 1 \leq i \leq l\}.$$

The set $X_{(f_1,\ldots,f_l)}$ is called a *subvariety* of $\mathbb{A}^n_k$, or an *affine variety*.

We now give some examples, starting with $n = 1$. In this case, a polynomial of degree d in a single variable has at most d roots, and X_f is a finite subset of $\mathbb{A}^1$.

Examples.

1. $d = 1$, $f(z) = z$. We get $X_f = \{0\}$, a single point.
2. $d = 2$, $f(z) = z^2 - 1$ .We get $X_f = \{-1, 1\}$, two distinct points.
3. $d = 2$, $f(z) = z^2$. We get $X_f = \{0\}$, but 0 "occurs" twice.
4. $d = 2$, $f(z) = z^2 + 1$. We get $X_f = \{-i, i\}$, over $\mathbb{C}$, but $\emptyset$ over $\mathbb{R}$.

Let us now consider the case $n = 2$. A hypersurface in $\mathbb{A}^2$ is called an *(affine) plane curve*.

Examples.

1. $f(z_1, z_2) = z_1$. We get the line of equation $z_1 = 0$.
2. $f(z_1, z_2) = z_1 z_2$. We get the two lines of equations $z_1 = 0$, $z_2 = 0$.
3. $f(z_1, z_2) = z_1^2 + z_2^2 - 1$. We get the circle of center $(0, 0)$ and radius 1.
3. $f(z_1, z_2) = z_1^2 + z_2^2 + 1$. In the real case, we get the empty set, in the complex case, an imaginary circle.
4. $\{f_1 = z_1,\ f_2 = z_2\}$. We get the point $\{(0, 0)\}$.
5. $\{f_1 = z_1 z_2,\ f_2 = z_2^2\}$. We get the line of equation $z_2 = 0$ plus the origin counted twice!

The last example suggests that a more sophisticated notion of variety is needed. There is such a notion based on *schemes*.

Problem 1. Given two sets of l polynomials

$$\{f_1(z_1, \ldots, z_n), \ldots, f_l(z_1, \ldots, z_n)\} \quad \text{and} \quad \{g_1(z_1, \ldots, z_n), \ldots, g_l(z_1, \ldots, z_n)\},$$

we say that the subvarieties $X_{(f_1,\ldots,f_l)}$ and $X_{(g_1,\ldots,g_l)}$ *are affinely equivalent* if there is an invertible affine map A (from $\mathbb{A}^n$ to $\mathbb{A}^n$) such that

$$g_i(A\mathbf{z}) = f_i(\mathbf{z}),$$

for all i, $1 \leq i \leq l$, where $\mathbf{z}$ denotes the column vector of coordinates $(z_1, \ldots, z_n)$. Observe that all hyperplanes are equivalent.

Classify the quadrics in $\mathbb{A}^n$ over $\mathbb{C}$, and over $\mathbb{R}$.

Hint: It may be easier to classify the quadrics in $\mathbb{P}^n$ over $\mathbb{C}$ (where $\mathbb{P}^n$ denotes the n-dimensional projective space, see Section 1.5), and over $\mathbb{R}$.

Relevant references include Mumford [27, 28], Fulton [9], Hartshorne [16], Griffiths and Harris [13], Shafarevich [34], Perrin [30], Harris [15]. For a comprehensive text on commutative algebra with a view toward algebraic geometry, see Eisenbud [6].

1.2 Some "Big" Topics: Outline Of The Course

Sheaves, cohomology, Hodge theory.

Vector bundles, Chern classes, Chern characters, Riemann-Roch Theorem.

Moduli spaces (and deformation theory).

Enumerative geometry $\approx$ intersection theory.

Group theoretic aspects (Borel-Weil-Bott Theorem).

Sources of examples:

curves,

surfaces,

toric varieties.

1.3 The Zariski Topology

Assume that we are in affine space $\mathbb{A}^n$ (over $\mathbb{C}$).

Definition 1.3.1 A *Zariski-closed subset* is any affine subvariety. A *Zariski-open subset* is the complement of a Zariski-closed subset.

It is easily verified that the Zariski-open subsets define a topology. It is a weak topology which is not Hausdorff (the open sets are "big"). For example, in $\mathbb{A}^1$, the closed sets are the finite subsets and $\mathbb{A}^1$. An affine variety $X \subseteq \mathbb{A}^n$ inherits a topology. This is called the *Zariski topology of* X.

Problem 2. Prove that any two (irreducible) algebraic plane curves in $\mathbb{C}^2$ are homeomorphic in the Zariski topology.

A topological space X is *reducible* if there exist two proper closed subsets X_1, X_2 such that $X = X_1 \cup X_2$. Otherwise, X is *irreducible*.

In general, if X is irreducible, it is connected, but the converse is false: see example 1.2 (in the case $n = 2$).

Given any affine subvariety $X \subseteq \mathbb{A}^n$, we can define a function ring $\mathbb{C}[X]$ as well as a sheaf $\mathcal{O}_X$. For example, for $X = \mathbb{A}^n$,

$$\mathbb{C}[\mathbb{A}^n] = \mathbb{C}[z_1, \ldots, z_n],$$

the ring of polynomial functions in n arguments.

If $X \subseteq \mathbb{A}^n$ is defined by the polynomials $(f_1, \ldots, f_l)$, we define $\mathbb{C}[X]$ as

$$\mathbb{C}[X] = \mathbb{C}[A^n]/(f_1, \ldots, f_l),$$

the quotient of $\mathbb{C}[A^n]$ by the ideal generated by $f_1, \ldots, f_l$.

However, this definition is wrong, because it depends on the choice of the polynomials defining X (compare Examples 1.5). This can be fixed in two ways:

Variant 1: Replace the ideal $(f_1, \ldots, f_l)$ by its radical (if the field is algebraically closed).

Variant 2: Enrich the notion of variety to keep track of extra information (use schemes).

The sheaf $\mathcal{O}_X$ is the assignment to each open subset U of the ring $\mathcal{O}_X(U)$ defined such that

$$\mathcal{O}_X(U) = \{f/g \mid f, g \in \mathbb{C}[X],\ g(u) \neq 0 \text{ for all } u \in U\}.$$

1.4 Presheaves and Sheaves

Let X be a topological space, and $\mathbf{G}$ a category of "gismos". For example, the gismos might be sets, groups, abelian groups, rings, commutative rings, algebras, etc.

We associate with X the category whose objects are the open sets of X, and whose morphisms are the inclusion maps $i\colon U \to V$, whenever $U \subseteq V$ and U and V are open sets in X. A fancy way to define a presheaf is to say that it is a contravariant functor from the topology of X (as a category) to $\mathbf{G}$. A more down-to-earth definition of a presheaf is as follows:

Definition 1.4.1 Given a topological space X and a category $\mathbf{G}$, a *presheaf* $\mathcal{F}$ *on* X *with values in* $\mathbf{G}$ consists of an assignement of an object $\mathcal{F}(U) \in \mathbf{G}$ to every open subset $U \subseteq X$, and of a morphism $\mathcal{F}(i)\colon \mathcal{F}(V) \to \mathcal{F}(U)$ to every inclusion $i\colon U \to V$, where $U \subseteq V$ are open sets in X, and so that

$$\begin{aligned}\mathcal{F}(j \circ i) &= \mathcal{F}(i) \circ \mathcal{F}(j),\\ \mathcal{F}(\mathrm{id}_U) &= \mathrm{id}_{\mathcal{F}(U)},\end{aligned}$$

for every pair of inclusions $i\colon U \to V$ and $j\colon V \to W$, where $U \subseteq V \subseteq W$, and $\mathrm{id}_U\colon U \to U$ is the identity.

The morphisms $\mathcal{F}(i)\colon \mathcal{F}(V) \to \mathcal{F}(U)$ are also denoted by $\rho^V_U\colon \mathcal{F}(V) \to \mathcal{F}(U)$. Intuitively, they are restriction maps, if we view $\mathcal{F}(U)$ as a set of functions with domain U (which is often the case). The identity $\mathcal{F}(j \circ i) = \mathcal{F}(i) \circ \mathcal{F}(j)$ is expressed by

$$\rho^W_U = \rho^V_U \circ \rho^W_V.$$

Examples of presheaves.

1. The constant presheaf with value G, where $G \in \mathbf{G}$ is any fixed object.

2. Given a topological space G, we define $\mathcal{C}(G)$ as the presheaf that assigns the set $\mathcal{C}(G)(U)$ of all continuous functions from U to G to the open set $U \subseteq X$.

2.1. If $G = (\mathbb{R}, +, \text{usual metric topology})$, then $\mathcal{C}(G)$ is the presheaf of real-valued continuous functions on X. It is a sheaf of $\mathbb{R}$-algebras.

2.2. If $G = (\mathbb{R}, +, \text{trivial topology})$, then $\mathcal{C}(G)$ as defined in (2) is called the presheaf of *discontinuous* functions (all functions on X).

Remark: A more general definition of a presheaf is sometimes needed. Rather than using a topological space X as the source category, one can use an arbitrary category $\mathbf{B}$. Then, a presheaf is a contravariant functor from $\mathbf{B}$ to $\mathbf{G}$. This is an assignment of an objet $\mathcal{F}(a) \in \mathbf{G}$

to every object $a \in \mathbf{B}$, and of a morphism $\mathcal{F}(f)\colon \mathcal{F}(b) \to \mathcal{F}(a)$ to every morphism $f\colon a \to b$ in $\mathbf{B}$, such that $\mathcal{F}$ is functorial. An example of such presheaves is given by *étale presheaves*, where the étale "topology" is not a topology in the usual sense. We will come back to this topic later.

Definition 1.4.2 Given a topological space X and a category $\mathbf{G}$, a *morphism* $\varphi\colon \mathcal{F} \to \mathcal{G}$ *of presheaves* $\mathcal{F}$ *and* $\mathcal{G}$ consists of a family of morphisms $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, where U is any open subset $U \subseteq X$, such that

$$\varphi_U \circ \mathcal{F}(i) = \mathcal{G}(i) \circ \varphi_V,$$

for every inclusion $i\colon U \to V$, where $U \subseteq V \subseteq X$, or equivalently, so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathcal{F}(i)"'] & \mathcal{G}(V) \arrow[d, "\mathcal{G}(i)"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{G}(U) \end{tikzcd}$$

Morphisms of presheaves are composed in the obvious way. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ are morphisms, then $\psi \circ \varphi\colon \mathcal{F} \to \mathcal{H}$ is the morphism defined such that

$$(\psi \circ \varphi)_U = \psi_U \circ \varphi_U.$$

The kernel and the cokernel of a morphism of presheaves are presheaves defined in the "natural way" (see later).

Presheaves are usually used to keep track of local information assigned to a global object (the space X). It is usually desirable to use some consistent local information to recover some global information, but this requires a sharper notion, that of a sheaf. To avoid complications, we first define sheaves of sets. After that, we explain how to adapt the definition to define sheaves where the objects belong to more general categories such as groups, abelian groups, rings, modules, algebras, topological spaces, etc.

Definition 1.4.3 Given a topological space X a *sheaf* $\mathcal{F}$ *on* X is a set-valued presheaf so that for every open set $U \subseteq X$, for every open cover $(U_i)_{i\in I}$ of U (i.e., $U = \bigcup_{i\in I} U_i$ and the U_i are open sets in X), the following properties hold:

(1) (glueing condition) For any family $(f_i)_{i\in I}$ such that $f_i \in \mathcal{F}(U_i)$, if

$$\rho^{U_i}_{U_i\cap U_j}(f_i) = \rho^{U_j}_{U_i\cap U_j}(f_j) \quad \text{for all } i, j \in I,$$

then there is some $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$ for all $i \in I$.

(2) (identity condition) Given any $f, g \in \mathcal{F}(U)$, if

$$\rho^U_{U_i}(f) = \rho^U_{U_i}(g) \quad \text{for all } i \in I,$$

then $f = g$.

Another notation often used for $\mathcal{F}(U)$ is $\Gamma(U, \mathcal{F})$. An element of $\Gamma(U, \mathcal{F})$ is also called a *section over* U, and elements of $\Gamma(X, \mathcal{F})$ are called *global sections*. Also, given two open subset U, V, with $V \subseteq U$, for any $s \in \mathcal{F}(U)$, we may denote $\rho^U_V(s)$ by $s|V$.

Remarks:

(i) If $\mathcal{F}(U) = \emptyset$ for some open subset $U \subseteq X$, then $\mathcal{F}$ is the trivial sheaf such that $\mathcal{F}(U) = \emptyset$ for all open subsets $U \subseteq X$. Indeed, there is a restriction function $\rho^X_U: \mathcal{F}(X) \to \emptyset$, but the only function with range $\emptyset$ is the "empty function" with domain $\emptyset$, and thus $\mathcal{F}(X) = \emptyset$, and $\mathcal{F}$ is trivial. From now on, we rule out this possibility.

(ii) Many books define sheaves of (abelian) groups, and not sheaves of sets. Since a group always contains the identity element 0, the set $\mathcal{F}(U)$ is never empty. This is the case for Hartshorne [16], Griffiths and Harris [13], Forster [8], Miranda [26], Narasimham [29]. On the other hand, Mumford [28], Eisenbud and Harris [7], Perrin [30], and Shafarevich [35], define sheaves of sets. The book by MacLane and Moerdijk [21] contains an extensive treatment of sheaves. Remark (i) was found in Godement [11].

(iii) Assuming that $\mathcal{F}$ is not the trivial sheaf, conditions (1) and (2) apply to all families of open covers $(U_i)_{i\in I}$ of all open subsets U of X, including the case where $I = \emptyset$ and $U = \emptyset$. In this case, condition (2) implies that $\mathcal{F}(\emptyset)$ is a one-element set. In the case of groups, rings, etc., this is the set $\{0\}$.

(iv) An equivalent way of stating conditions (1) and (2) together is to use the maps

$$p: \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j) \quad \text{and} \quad q: \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

defined such that

$$p((f_i)_{i\in I}) = (\rho^{U_i}_{U_i\cap U_j}(f_i))_{(i,j)\in I\times I} \quad \text{and} \quad q((f_i)_{i\in I}) = (\rho^{U_j}_{U_i\cap U_j}(f_j))_{(i,j)\in I\times I}.$$

Then, for every open subset U of X and every open cover $(U_i)_{i\in I}$ of U, the map

$$e: \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

defined such that

$$e(f) = (\rho^U_{U_i}(f))_{i\in I}$$

must be an equalizer of the maps p and q:

$$\mathcal{F}(U) \xrightarrow{e} \prod_{i\in I} \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j).$$

This means that $p \circ e = q \circ e$, and that for every map $g\colon A \to \prod_{i\in I} \mathcal{F}(U_i)$ such that $p \circ g = q \circ g$, there is a unique map $\widehat{g}\colon A \to \mathcal{F}(U)$ such that

$$g = e \circ \widehat{g}.$$

Observe that a product over the empty family $I = \emptyset$ is a one-element set, which corresponds to the fact that $\mathcal{F}(\emptyset)$ is a one-element set. The equalizer condition is from MacLane and Moerdijk [21]. Mumford [28] uses a slight different formulation (exact sequence) equivalent to the present one.

Definition 1.4.3 can be generalized to sheaves of sets with some additional structure, such as groups, rings, modules, algebras, etc. The only difference is that the restriction maps $\rho^V_U\colon \mathcal{F}(V) \to \mathcal{F}(U)$ must be morphisms of the category $\mathbf{G}$ of objects under consideration (sets with structure). For example, if the objects of $\mathbf{G}$ are groups, the restriction maps are group homomorphisms.

Remark: Using the equalizer condition for sheaves, it is possible to give a more formal definition of a sheaf with values in a category $\mathbf{G}$, assuming that $\mathbf{G}$ has products, including a terminal object. Indeed, the maps, p, q, e can be expressed as product maps

$$p = \prod_{(i,j)\in I\times I} \rho^{U_i}_{U_i\cap U_j}, \qquad q = \prod_{(i,j)\in I\times I} \rho^{U_j}_{U_i\cap U_j}, \qquad e = \prod_{i\in I} \rho^{U}_{U_i},$$

and we can define a sheaf $\mathcal{F}$ on X with values in $\mathbf{G}$ as a presheaf from X to $\mathbf{G}$ such that

$$\mathcal{F}(U) \xrightarrow{e} \prod_{i\in I} \mathcal{F}(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram for every open subset U of X and every open cover $(U_i)_{i\in I}$ of U. For more details, see MacLane and Moerdijk [21].

Examples of sheaves.

1. If $\pi\colon Y \to X$ is a continuous surjective map between topological spaces, it is not hard to prove that the *presheaf $\mathcal{S}$ of (continuous) sections* is indeed a sheaf, where

$$\mathcal{S}(U) = \{s\colon U \to \pi^{-1}(U) \mid \pi \circ s = \mathrm{id}\}.$$

   Observe that Example 2 (of presheaves) corresponds to the case where $Y = X \times G$, and π is the first projection.

2. In particular, if G is given the discrete topology, $Y = X \times G$, and π is the projection onto X, the sheaf $\mathcal{S}$ of sections is the sheaf of locally constant functions.

The constant presheaf is not a sheaf! Indeed, the glueing condition may fail when X is disconnected.

3. The kernel of a morphism of sheaves is a sheaf.

The cokernel of a morphism of sheaves is not a sheaf. In fact, the image of a sheaf morphism is not a sheaf in general.

Problem 3. Define the kernel of a sheaf morphism and prove that it is a sheaf. Define the image of a sheaf morphism and prove that it is generally not a sheaf.

Remark: A more general definition of a sheaf using a category $\mathbf{B}$ rather than a topological space X can also be given, as in the case of presheaves. However, a more general concept of cover is needed (see MacLane and Moerdijk [21]).

Sheaves yield a uniform way of defining various kinds of manifolds, bundles, and algebraic varieties. For this, we need to define ringed spaces and maps between ringed spaces. First, we note that a sheaf induces a subsheaf on every open subset.

Given a sheaf $\mathcal{F}$ on a space X, for any open subset U of X, we define the restriction $\mathcal{F}|U$ of the sheaf $\mathcal{F}$ to U in the obvious way, i.e., $(\mathcal{F}|U)(V) = \mathcal{F}(V)$ for every open subset $V \subseteq U$.

Definition 1.4.4 A *ringed space* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings, often denoted by $\mathcal{O}_X$, and called the *structure sheaf*.

We now need to define the notion of map between two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. The basic idea is that such a morphism f is a continuous map between the underlying topological spaces that "pulls back" the sheaf of functions on Y to the sheaf of functions on X. For simplicity, let us first assume that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are both sheaves of functions respectively on X and Y. Let $f\colon X \to Y$ be a continuous function. Given any function $h \in \mathcal{O}_Y(V)$, we denote $(h \circ f)|f^{-1}(V)$ by f^*h. Then, we say that $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a *morphism of ringed spaces* if the following condition holds:

For every open subset V of Y,

$$\text{if} \quad h \in \mathcal{O}_Y(V) \quad \text{then} \quad f^*h \in \mathcal{O}_X(f^{-1}(V)).$$

Observe that for every open subset $V \subseteq Y$, the assignment $h \mapsto f^*h$ defines a map

$$f_V^*\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)),$$

and f_V^* is a ring homomorphism. Also note that the maps f_V^* make the following diagrams commute for all pairs of open subsets (U, V) such that $U \subseteq V \subseteq Y$:

$$\begin{array}{ccc}
\mathcal{O}_Y(V) & \xrightarrow{f_V^*} & \mathcal{O}_X(f^{-1}(V)) \\
{\scriptstyle \rho_U^V}\downarrow & & \downarrow{\scriptstyle \rho_{f^{-1}(U)}^{f^{-1}(V)}} \\
\mathcal{O}_Y(U) & \xrightarrow[f_U^*]{} & \mathcal{O}_X(f^{-1}(U))
\end{array}$$

where ρ_U^V and $\rho_{f^{-1}(U)}^{f^{-1}(V)}$ are the obvious restriction maps. We say that (f, f^*) is a *morphism of ringed spaces*. When $\mathcal{O}_X$ and $\mathcal{O}_Y$ are not necessarily sheaves of functions, we need to enforce the above conditions using some suitable morphism of sheaves. For this, we need the notion of direct image sheaf.

Given any continuous maps $f\colon X \to Y$ between two topological spaces X, Y, for any sheaf $\mathcal{F}$ on X, we define $f_*\mathcal{F}$, called the *direct image of $\mathcal{F}$ under f*, as the sheaf on Y defined such that

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for every open subset $V \subseteq Y$. It is easily verified that $f_*\mathcal{F}$ is indeed a sheaf. Then, given two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are not necessarily sheaves of functions, we define a morphism of ringed spaces as follows:

Definition 1.4.5 A *morphism of ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ as a pair (f, g), where $f\colon X \to Y$ is a continuous map and $g\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves, and where each $g_V\colon \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$ is a homomorphism of rings.

Morphisms of ringed spaces are composed in the obvious way, i.e, $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$, and a morphism of ringed spaces is an isomorphism if it has an inverse, i.e., there is a morphism (f', g') from $(Y, \mathcal{O}_Y)$ to $(X, \mathcal{O}_X)$ such that $(f', g') \circ (f, g) = (\mathrm{id}, \mathrm{id})$ and $(f, g) \circ (f', g') = (\mathrm{id}, \mathrm{id})$. Given a ringed space $(X, \mathcal{O}_X)$, for every open subset $U \subseteq X$, it is clear that $(U, \mathcal{O}_X|U)$ is a ringed space. For more details, see Hartshorne [16], Chapter II, Section 2.

We can now use the above definition to define various kinds of manifolds. The general idea is that a ringed space $(X, \mathcal{O}_X)$ is a certain type of manifold if it is locally isomorphic to some other ringed space of the required type. The sheaf $\mathcal{O}_X$ on X defines the "good " functions on X.

Definition 1.4.6 Given two ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, we say that $(X, \mathcal{O}_X)$ *is locally isomorphic to* $(Y, \mathcal{O}_Y)$ if for every $x \in X$, there is some open subset, U, of X, with $x \in U$, and some open subset, V, of Y, so that the ringed spaces $(U, \mathcal{O}_X|U)$ and $(V, \mathcal{O}_Y|V)$ are isomorphic.

We can now define various kinds of manifolds.

Definition 1.4.7 A *continuous manifold* is a ringed space M (with a sheaf of real algebras) which is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}(\mathbb{R}))$, where $\mathcal{C}(\mathbb{R})$ is the sheaf of algebras of continuous functions on $\mathbb{R}^n$.

A similar definition yields smooth and complex analytic manifolds.

Definition 1.4.8 A *smooth manifold* is a ringed space M (with a sheaf of real algebras) which is locally isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty(\mathbb{R}))$, where $\mathcal{C}^\infty(\mathbb{R})$ is the sheaf of algebras of smooth functions on $\mathbb{R}^n$.

Definition 1.4.9 A *complex analytic manifold* is a ringed space M (with a sheaf of complex algebras) which is locally isomorphic to $(\mathbb{C}^n, \mathrm{Hol}(\mathbb{C}^n))$, where $\mathrm{Hol}(\mathbb{C}^n)$ is the sheaf of algebras of holomorphic functions on $\mathbb{C}^n$.

We can define a *vector bundle* as a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the ringed space of sections from X to $X \times V$, where V is a fixed vector space.

Finally, an *algebraic variety* is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the ringed space of functions on an some affine variety.

Remark: There is an uncountable family of nonisomorphic affine varieties.

1.5 Projective Spaces, Projective Varieties

We begin by defining projective spaces. The (complex) projective space $\mathbb{P}^n$ is the quotient of the set $\mathbb{A}^{n+1} - \{0\}$ by the equivalence relation defined such that

$$u \simeq v \quad \text{iff} \quad \exists \lambda \in \mathbb{C} - \{0\},\, v = \lambda u.$$

We will often denote the equivalence class of a point $(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}$ as $[x_0, \ldots, x_n]$.

We define the open cover $\{U_0, \ldots, U_n\}$ of $\mathbb{P}^n$ as follows:

$$U_i = \{[x_0, \ldots, x_n] \in \mathbb{P}^n \mid x_i \neq 0\}.$$

The map

$$[x_0, \ldots, x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

is a bijection between U_i and $\mathbb{A}^n$. It is a coordinate map on U_i.

When $n = 1$, we have two open sets $U_0 = \{[x_0, x_1] \mid x_0 \neq 0\}$ and $U_1 = \{[x_0, x_1] \mid x_1 \neq 0\}$. The coordinates maps on U_0 and U_1 are $z = x_0/x_1$ and $w = x_1/x_0$. Observe that $U_0 \cap U_1 = \mathbb{A}^1 - \{0\}$. The glueing condition is $z = w^{-1}$. The space $\mathbb{P}^1$ is topologically the Riemann sphere S^2. It is covered by two copies of $\mathbb{A}^1$.

There is another variety covered by two copies U_0, U_1 of $\mathbb{A}^1$, with $U_0 \cap U_1 = \mathbb{A}^1 - \{0\}$: use the glueing condition $z = w$. In this case, we get a non Hausdorff space consisting of a copy of $\mathbb{A}^1 - \{0\}$ and the origin counted twice!

The definition of a projective subvariety is similar to the definition of an affine subvariety, except that homogeneous polynomials are used.

Given a set of l homogeneous polynomial $\{f_1, \ldots, f_l\}$ with $f_i \in k[z_0, \ldots, z_n]$, we define $X_{(f_1,\ldots,f_l)}$ as

$$X_{(f_1,\ldots,f_l)} = \{[x_0, \ldots, x_n] \in \mathbb{P}^n \mid f_i(x_0, \ldots, x_n) = 0,\, 1 \leq i \leq l\}.$$

The set $X_{(f_1,\ldots,f_l)}$ is called a *projective subvariety* of $\mathbb{P}^n$, or a *projective variety*.

It is necessary to check that the defintion of $X_{(f_1,\ldots,f_l)}$ does not depend on the choice of representative $(x_0,\ldots,x_n) \in \mathbb{C}^{n+1}$ of the point $[x_0,\ldots,x_n] \in \mathbb{P}^n$. However, since each $f_i \in k[z_0,\ldots,z_n]$ is homogeneous, say of degree d_i, for any $\lambda \in \mathbb{C}$,

$$f_i(\lambda x_0,\ldots,\lambda x_n) = \lambda^{d_i} f_i(x_0,\ldots,x_n),$$

and thus, for every $\lambda \neq 0$,

$$f_i(\lambda x_0,\ldots,\lambda x_n) = 0 \quad \text{iff} \quad f_i(x_0,\ldots,x_n) = 0.$$

Again, relevant references include Mumford [27, 28], Fulton [9], Hartshorne [16], Griffiths and Harris [13], Shafarevich [34], Perrin [30], Harris [15], and also Samuel [31] and Berger [1] for a more extensive treatment of projective geometry.

1.6 Duality in Projective Spaces, Quadrics

We first discuss duality in $\mathbb{P}^2$ informally. We will return to this more rigorously. Recall that the equation of a line in homogeneous coordinates is of the form

$$ux_0 + vx_1 + wx_2 = 0.$$

Points in $\mathbb{P}^2$ are defined by homogeneous coordinates (a_0, a_1, a_2). We can view (u, v, w) as homogeneous coordinates of the line of equation

$$ux_0 + vx_1 + wx_2 = 0.$$

Since for any $\lambda \neq 0$, the equation

$$\lambda ux_0 + \lambda vx_1 + \lambda wx_2 = 0$$

defines the same line, it is indeed legitimate to view (u, v, w) as homogeneous coordinates of a line. Then, we have a duality between points and lines in $\mathbb{P}^2$. We denote the projective space of lines as $\mathbb{P}^{2^*}$. The duality is really defined by the pairing

$$\cdot\colon \mathbb{P}^2 \times \mathbb{P}^{2^*} \to \mathbb{C}$$

given by

$$(a_0, a_1, a_2) \cdot (u, v, w) = ua_0 + va_1 + wa_2.$$

This is the *incidence relation* (the point (a_0, a_1, a_2) belongs to the line (u, v, w), or equivalently, the line (u, v, w) passes through the point (a_0, a_1, a_2)).

Duality can be used to translate properties about points to properties about lines. For example, we have the following correspondences:

points	lines
There exists a unique line through two distinct points	There exists a unique point of intersection of two distinct lines
There exists a unique conic through five general points	There exists a unique conic tangent to five general lines

Duality can be generalized to $\mathbb{P}^n$. In this case, note that a hyperplane is defined by an equation of the form

$$b_0x_0 + \cdots + b_nx_n = 0.$$

Thus, points correspond to hyperplanes. More generally, a projective subspace $\mathbb{P}^k$ of dimension k corresponds to the projective subspace $\mathbb{P}^{n-k-1}$ of dimension $n-k-1$.

We can also define the dual of a non-singular quadric hypersurface Q. A *(projective) quadric hypersurface* is the subvariety Q of $\mathbb{P}^n$ defined by a homogeneous equation of the form

$$X^\top AX = 0,$$

where A is a symmetric matrix, and where $X = (x_0, x_1, \ldots, x_n)$. The quadric is *nonsingular* if A is invertible.

Then, the dual Q^* of Q is the set of hyperplanes tangent to Q. It turns out to be the non-singular quadric defined by the inverse A^{-1} of A. Equivalently, it is the quadric defined by the matrix $\bigwedge^n A$ of cofactors of A. The advantage of this second formulation is that is works when A is singular.

More generally, given a subvariety $X \subseteq \mathbb{P}^n$, we can define the dual subvariety X^*. This requires either defining the tangent space $T_x(X)$ to X at a point x, or the notion of $H \cap X$ being singular, where H is a hyperplane. These definitions will be given later on. Losely speaking, we define

$$X_1^* = \{H \mid H \cap X \text{ is singular}\},$$

and

$$X_2^* = \{H \mid T_x(X) \subseteq X, \text{ for some } x \in X\}.$$

The two definitions do not yield the same result in general. However, in both cases, points correspond to hyperplanes and the projective subspace $\mathbb{P}^k$ corresponds to the projective subspace $\mathbb{P}^{n-k-1}$.

If X is a quadric defined by the symmetric matrix A, then X^* is also a quadric, even if X is singular. In fact, if X is singular, X^* is a nonsingular quadric contained in a projective subspace.

Let us now classify the quadrics in $\mathbb{P}^n$ over $\mathbb{C}$. If Q_1 and Q_2 are two quadrics defined by some symmetric matrices A and B, since the equations of the quadrics are

$$X^\top AX = 0 \quad \text{and} \quad X^\top BX = 0,$$

Q_1 and Q_2 are equivalent iff there is a bijective linear map L (given by a nonsingular matrix) such that

$$A = L^\top BL.$$

Since a symmetric bilinear form φ can be expressed by a diagonal matrix over an orthogonal basis w.r.t. φ, any quadric is equivalent to a quadric defined by a diagonal matrix D. Furthermore, since we are over $\mathbb{C}$, by taking square roots, we can assume that the quadric is defined by a diagonal matrix consisting of 1's and 0's. Thus, the quadrics in $\mathbb{P}^n$ over $\mathbb{C}$ are classified by rank.

For example, in $\mathbb{P}^1$, we get:

rank 2: two distinct points

rank 1: a double point.

In $\mathbb{P}^2$, we get:

rank 3: a nonsingular conic

rank 2: two distinct (intersecting) lines

rank 1: a double line.

In $\mathbb{P}^3$, we get:

rank 4: a nonsingular quadric

rank 3: a cone

rank 2: two planes intersecting in a line

rank 1: a double plane.

If Q is a quadric in $\mathbb{P}^n$ over $\mathbb{R}$, we can no longer take square roots of negative numbers, and the diagonal matrix defining Q has $+1$, -1, and 0, as entries. By Sylvester's inertia theorem, the number n_+ of $+1$'s and the number n_- of -1's are invariant. Then, the quadrics in $\mathbb{P}^n$ over $\mathbb{R}$ are classified by the pairs (n_+, n_-) (called type, or signature), where (n_-, n_+) and (n_+, n_-) are considered equivalent, since the equations

$$z_1^2 + \cdots + z_{n_+}^2 - z_{n_++1}^2 - \cdots - z_{n_++n_-}^2 = 0$$

and

$$-z_1^2 - \cdots - z_{n_+}^2 + z_{n_++1}^2 + \cdots + z_{n_++n_-}^2 = 0$$

define the same quadric.

For example, in $\mathbb{P}^1$ (also denoted as $\mathbb{RP}^1$), we get:

$(2, 0)$, for the equation $x^2 + y^2 = 0$, empty

$(1, 1)$, for the equation $x^2 - y^2 = 0$, two distinct lines

$(1, 0)$, for the equation $x^2 = 0$, a double point

$(0, 0)$, for the equation $0 = 0$, all of $\mathbb{P}^1$.

In $\mathbb{P}^2$ (also denoted as $\mathbb{RP}^2$), we get:

$(3, 0)$, for the equation $x^2 + y^2 + z^2 = 0$, empty conic

$(2, 1)$, for the equation $x^2 + y^2 - z^2 = 0$, a conic

$(2, 0)$, for the equation $x^2 + y^2 = 0$, a point

$(1, 1)$, for the equation $x^2 - y^2 = 0$, two distinct lines

$(1, 0)$, for the equation $x^2 = 0$, a double line

$(0, 0)$, for the equation $0 = 0$, all of $\mathbb{P}^2$.

To classify the quadrics in $\mathbb{A}^n$, we use the fact that there is a bijection

$$\mathbb{P}^n \approx \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

Thus, we need to take into account the intersection of a (projective) quadric Q with $\mathbb{P}^{n-1}$ to classify the affine quadrics.

1.7 More On Duality in Projective Spaces, The Dual Variety

Given a vector space V of dimension n, we let V^* denote the vector space of all linear forms on V (the linear maps $\varphi\colon V \to \mathbb{C}$), called the *dual space of* V. The projective spaces induced by V and V^* are denoted as $\mathbf{P}(V)$ and $\mathbf{P}(V^*)$. The projective space $\mathbf{P}(V^*)$ is also denoted as $\mathbf{P}(V)^*$. Observe that the points of $\mathbf{P}(V)^*$ can be viewed as hyperplanes in $\mathbf{P}(V)$ (or as hyperplanes through the origin in V). Then, we have a duality between projective subspaces of $\mathbf{P}(V)$ and projective subspaces of $\mathbf{P}(V^*)$, such that points in $\mathbf{P}(V)$ correspond to hyperplanes in $\mathbf{P}(V^*)$. This duality is induced by the pairing

$$\langle -, - \rangle\colon V \times V^* \to \mathbb{C},$$

where

$$\langle u, \varphi \rangle = \varphi(u).$$

In coordinates, using a basis on V and its dual basis in V^*, the pairing is given by

$$\langle (a_0, \ldots, a_n), (b_0, \ldots, b_n) \rangle = a_0 b_0 + \cdots + a_n b_n,$$

where $(a_0, \ldots, a_n)$ are the coordinates of a vector in V and $(b_0, \ldots, b_n)$ are the coordinates of a linear form in V^*. Now, given any subspace U of V and any subspace W of V^*, we define the subspaces $U^\perp \subseteq V^*$ and $W^\perp \subseteq V$ as follows:

$$\begin{aligned} U^\perp &= \{f \in V^* \mid f(u) = 0, \text{ for all } u \in U\}, \\ W^\perp &= \{u \in V \mid f(u) = 0, \text{ for all } f \in W\}. \end{aligned}$$

It is immediately verified that the maps

$$U \mapsto U^\perp \quad \text{and} \quad W \mapsto W^\perp$$

are bijections mapping subspaces of V to subspaces of V^* (and conversely). Also, if $\dim(V) = n+1$ and $\dim(U) = k$, then $\dim(U^\perp) = n+1-k$. The above maps induce a bijection between projective subspaces of $\mathbf{P}(V)$ and projective subspaces of $\mathbf{P}(V^*)$, such that if $U \subseteq \mathbf{P}(V)$ and $\dim(U) = k$, then $\dim(U^\perp) = n-k-1$ (assuming that $\dim(V) = n+1$). For instance, points in $\mathbf{P}(V)$ ($k = 0$) correspond to hyperplanes in $\mathbf{P}(V^*)$. Since points in $\mathbf{P}(V^*)$ correspond to hyperplanes in $\mathbf{P}(V)$, duality can be viewed as mapping a projective subspace $U \subseteq \mathbf{P}(V)$ to the set of hyperplanes $H \subseteq \mathbf{P}(V)$ containing U, and mapping a linear system W of hyperplanes (the set of hyperplanes associated with the (nonnull) linear forms in a subspace $W \subseteq \mathbf{P}(V^*)$) to the intersection of the hyperplanes in W.

For example, when $n = 2$, a point $p \in \mathbb{P}^2$ corresponds to the pencil of lines (in $\mathbb{P}^2$) passing through p. When $n = 3$, a point $p \in \mathbb{P}^3$ corresponds to the set of planes passing through p, and a line $L \in \mathbb{P}^3$ corresponds to the set of planes (in $\mathbb{P}^3$) containing L, called a pencil of planes. For more details, see Samuel [31] and Berger [1].

There are two ways of defining the dual of a closed variety $X \subseteq \mathbb{P}^n$. We can define

$$X_1^* = \text{closure}(\{H \in \mathbb{P}^{n*} \mid H \cap X \text{ is singular}\}),$$

and

$$X_2^* = \text{closure}(\{H \in \mathbb{P}^{n*} \mid T_x(X) \subseteq X, \text{ for some nonsingular point } x \in X\}).$$

When X is singular, X_1^* and X_2^* could be different. For example, assume that X is a *cone* over some variety $Y \subseteq \mathbb{P}^{n-1}$. In coordinates, this cone has for vertex the point $(1, 0, \ldots, 0)$, and Y is contained in the hyperplane H of equation $x_0 = 0$. Any point on X of homogeneous coordinates $(a_0, \ldots, a_n)$ belongs to a unique line joining the point $(1, 0, \ldots, 0)$ to a point $(0, a_1, \ldots, a_n)$ in H. If Y is defined by equations

$$\{f_1(x_1, \ldots, x_n) = 0, \ldots, f_l(x_1, \ldots, x_n) = 0\},$$

where each f_j is a homogeneous polynomial in the variables $x_1, \ldots, x_n$, then the equations for X are also

$$\{f_1(x_1, \ldots, x_n) = 0, \ldots, f_l(x_1, \ldots, x_n) = 0\},$$

but now, the f_j are viewed as polynomials in $x_0, x_1, \ldots, x_n$. Assuming for simplicity that Y is nonsingular and that it is a hypersurface, the only singular point of the cone X is its vertex $(1, 0, \ldots, 0)$.

As an illustration, if $n = 3$ and Y is the conic of equation $x^2 + y^2 - z^2 = 0$ contained in the hyperplane $w = 0$, the cone X based on Y is the set of points

$$X = \{(w, x, y, z) \in \mathbb{P}^3 \mid x^2 + y^2 - z^2 = 0\}.$$

If the tangent space T_yY to Y at $y = (y_1, \ldots, y_n)$ is determined by $(b_1, \ldots, b_n)$, then the tangent space at $(y_0, \ldots, y_n)$ is determined by $(0, b_1, \ldots, b_n)$. Therefore,

$$X_1^* = \{(0, b_1, \ldots, b_n) \mid (b_1, \ldots, b_n) \in Y^*\}.$$

Observe that X_2^* contains X_1^*, but it also contains all hyperplanes through $(1, 0, \ldots, 0)$, namely, the entire hyperplane $\mathbb{P}^{n-1}$. Thus, $X_2^* = \mathbb{P}^{n-1}$, too much information is lost!

1.8 Parameter Spaces

One of the themes of algebraic geometry is to parameterize various families of varieties. Unlike manifolds, where the parameter space of a family of manifolds is generally far from being a manifold, the parameter space of a family of varieties is also algebraic.

The simplest example of a parameter space is $\mathbb{P}^n$ itself, as a parameter space for the set of all points (in $\mathbb{P}^n$). Lines in $\mathbb{P}^2$ correspond to points in $\mathbb{P}^{2^*}$, and thus, $\mathbb{P}^{2^*}$ is a parameter space for the lines in $\mathbb{P}^2$.

Can we parameterize pairs of points (say, in $\mathbb{A}^n$)? What does this mean? Do we mean pairs of distinct points, or sets of two points? We will need to consider these various cases. But first, we give a more precise definition of a parameter space, followed by some examples.

Definition 1.8.1 Given a projective variety X and a class $\mathcal{C}$ of subvarieties of X, a *parameter space for* $\mathcal{C}$ is a variety P together with a subvariety $C \subseteq P \times X$, such that the following properties hold:

1. For every $p \in P$, $f^{-1}(p)$ is a variety in $\mathcal{C}$, where $f\colon C \to P$ is the first projection.
2. For every variety $Y \in \mathcal{C}$, there is a unique $p \in P$ such that $f^{-1}(p) = Y$.

Later on, we will encounter more general parameter spaces for which $\mathcal{C}$ is some class of algebro-geometric objects, not necessarily subvarieties of a fixed X. Let us give some examples:

1. If $\mathcal{C}$ is the class of points in a variety X, then $C = P = X$ and $f = \mathrm{id}$.

2. Is there a reasonable parameter space for pairs of points?

3. A parameter space for lines in the plane $\mathbb{P}^2$ is given by $(C, \mathbb{P}^{2^*})$, where $C \subseteq \mathbb{P}^2 \times \mathbb{P}^{2^*}$ is the "incidence correspondence" defined by

$$C = \{(a, b) \in \mathbb{P}^2 \times \mathbb{P}^{2^*} \mid \langle a, b \rangle = 0\},$$

and where f is the second projection $\pi_2 \colon \mathbb{P}^2 \times \mathbb{P}^{2^*} \to \mathbb{P}^{2^*}$.

4. Lines in $\mathbb{P}^3$ are parameterized by the *Grassmannian* variety $\mathbf{G}(2,4)$ (also denoted as $\mathbf{PG}(1,3)$).

5. The conics in $\mathbb{P}^2$ are parameterized by $P = \mathbb{P}^5$.

6. How do we parameterize the singular conics in $\mathbb{P}^2$?

   Since a conic defined by a 3×3-symmetric matrix A is singular iff $\det(A) = 0$, the set of singular conics is an algebraic subvariety of $\mathbb{P}^5$. Since it is defined by a single equation of degree 3, it is a variety of dimension 4 and degree 3 (see the Section 1.9 for precise definitions of dimension and degree). Intuitively, if $X \subseteq \mathbb{P}^n$ is a k-dimensional variety, for any "general" subspace $L \subseteq \mathbb{P}^n$ of dimension $n - k$, we expect $X \cap L$ to consist of a finite number of points, independently of L (if L is general), and this number is the degree of X.

7. How do we parameterize the double lines in $\mathbb{P}^2$?

   Double lines correspond to singular conics for which the matrix A has rank 1. They are also the conics whose equation is a perfect square

$$(b_0 x_0 + b_1 x_1 + b_2 x_2)^2 = 0.$$

   Thus, the double lines correspond to the image of the map from $\mathbb{P}^2$ to $\mathbb{P}^5$ defined by

$$(b_0, b_1, b_2) \mapsto (b_0^2, b_1^2, b_2^2, b_0 b_1, b_0 b_2, b_1 b_2).$$

   The corresponding subvariety of $\mathbb{P}^5$ is known as a *Veronese variety*.

More generally, given any two integers $n, k \geq 1$, we have the *Veronese embedding* $V \colon \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$V(a_0, \ldots, a_n) = (a_0^k, \ldots, a_0^{k_0} \cdots a_n^{k_n}, \ldots, a_n^k),$$

where $k_0 + \cdots + k_n = k$, i.e., $a_0^{k_0} \cdots a_n^{k_n}$ is a monomial of degree k in $a_0, \ldots, a_n$. It is easily shown that there are $\binom{n+k}{k}$ such monomials, and thus,

$$N = \binom{n+k}{k} - 1.$$

A coordinate-free definition of the Veronese embedding can also be given. Given a vector space V of dimension $n+1$, letting $\mathrm{Sym}^d(V)$ denote the vector space of symmetric d-tensors (over V), the Veronese embedding is the projective map (also called projectivity) induced by the linear map from V to $\mathrm{Sym}^d(V)$ given by

$$u \mapsto \underbrace{u \odot \cdots \odot u}_{k} = u^{\bigodot k},$$

where we use the symbol $\bigodot$ instead of $\otimes$ to denote symetric tensors.

Problem 4. The parameter space of all plane cubic curves can be identified with the projective space $\mathbb{P}^9$. This has a natural stratification by singularity type of the cubic. (This is analogous to the stratification of the parameter space of conics ($= \mathbb{P}^5$)) by the singularity type of the conic, or equivalently by the rank of the corresponding symmetric matrix.)

(a) Identify nine subsets of $\mathbb{P}^9$, each parametrizing cubics with a particular singularity type. (E.g.: there is an open stratum of non-singular cubics, a codimension 1 stratum of cubics which intersect themselves once transversally, a codim 2 locus of cubics with a cusp,...)

(b) Find the dimension of each stratum.

(c) Figure out which strata are in the closure of which other strata.

(d) Whenever possible, find the degree (as projective variety in $\mathbb{P}^9$) of the closure of each stratum. (Some of these are hard.)

(e) What is the degree of the dual variety of a plane cubic, and what does it have to do with the rest of this problem.

(f) For each stratum, are all the curves parametrized by the points of that stratum isomorphic to each other (via automorphisms of $\mathbb{P}^2$)?

1.9 Properties of Varieties

Given an algebraic variety X, typically, definitions such as dimension, degree, singularity at a point, etc., are of the form:

X is "blah" if it is locally "blah" at every $x \in X$.

This means that there is a Zariski open set U such that $x \in U$ and such that "blah" holds for $X \cap U$. We begin with the definition of dimension of an affine variety.

Dimension:

Definition 1.9.1 Given an affine variety $X \subseteq \mathbb{A}^n$, for any $x \in X$, we say that X *has dimension at least k at x* if there is a Zariski-open subset $U \subseteq \mathbb{A}^n$, with $x \in U$, and $n-k$ polynomials $f_1(x_1, \ldots, x_n), \ldots, f_{n-k}(x_1, \ldots, x_n)$, such that

$$X \cap U = X_{(f_1,\ldots,f_{n-k})} \cap U.$$

In other words, X is locally defined by $n-k$ equations at x. We say that X *has dimension k at x* if k is the largest integer such that X has dimension at least k at x. We say that X *has dimension k* if it has dimension k at every point $x \in X$.

As usual, a definition about affine varieties is extended to abstract varieties using the notion of local isomorphism.

Definition 1.9.2 Given an algebraic variety X, we say that X *has dimension k* iff X is locally isomorphic to an affine variety Y which has dimension k.

For example, $\mathbb{P}^n$ has dimension n. A curve in $\mathbb{A}^2$ or $\mathbb{P}^2$ has dimension 1, since it is defined by a single equation. More generally, a hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ has dimension $n-1$.

One should be aware that the open subset U involved in Definition 1.9.1 may depend on the point $x \in X$, as shown by the next example.

Consider the affine curve $T \subseteq \mathbb{A}^3$ defined parametrically as

$$T = \{(t, t^2, t^3) \mid t \in \mathbb{C}\},$$

and its projective closure $\overline{T}$ defined as

$$\overline{T} = \{(s^3, s^2t, st^2, t^3) \mid (s,t) \in \mathbb{P}^1\},$$

where in the above, (s,t) are viewed as homogeneous coordinates of points in $\mathbb{P}^1$. The curve $\overline{T}$ (and T) is called the *twisted cubic*.

We claim that $\dim(T) = \dim(\overline{T}) = 1$.

Assuming that the homogeneous coordinates in $\mathbb{P}^3$ are (x,y,z,w), it is easily shown that the homogeneous ideal associated with $\overline{T}$ is generated by the three quadrics

$$\begin{aligned} Q_1 : &\quad xz - y^2, \\ Q_2 : &\quad yw - z^2, \\ Q_3 : &\quad xw - yz. \end{aligned}$$

Then, it is easily verified that

$$\begin{aligned} Q_1 \cap Q_2 &= \overline{T} \cup X_{(y=0,z=0)}, \\ Q_1 \cap Q_3 &= \overline{T} \cup X_{(x=0,y=0)}, \\ Q_2 \cap Q_3 &= \overline{T} \cup X_{(z=0,w=0)}. \end{aligned}$$

Thus, $Q_i \cap Q_j$ contains $\overline{T}$ and some additional line L_{ij} (where $1 \leq i,j \leq 3$, and $i \neq j$). Since

$$\overline{T} = Q_1 \cap Q_2 \cap Q_3,$$

every point $x \in \overline{T}$ belongs to the complement of one of the lines L_{ij}, and thus, this complement is an open set U such that

$$\overline{T} \cap U = Q_i \cap Q_j \cap U,$$

and locally, $\overline{T}$ is indeed defined by two equations, so that the dimension is $3 - 2 = 1$. Observe that there is no global neighborhood U that works for all $x \in \overline{T}$.

Degree:

Given a projective variety $X \subseteq \mathbb{P}^n$, if X has dimension k, it can be shown that $X \cap L$ consists of a constant finite number of points for any "general" subspace $L \subseteq \mathbb{P}^n$ of dimension $n - k$. This is a form of *Bertini's Theorem*, see Griffith and Harris [13].

Definition 1.9.3 Given a projective variety $X \subseteq \mathbb{P}^n$, if X has dimension k, we define the *degree of* X to be the number of intersection points with X of any general linear subspace $L \subseteq \mathbb{P}^n$ of dimension $n - k$ with X.

Singular and Nonsingular Points:

Definition 1.9.4 Given an affine variety $X \subseteq \mathbb{A}^n$, for any $x \in X$, if X has dimension k at x, we say that X *is nonsingular at* x if there is a Zariski-open subset $U \subseteq \mathbb{A}^n$ with $x \in U$, $n - k$ polynomials $f_1(x_1, \ldots, x_n), \ldots, f_{n-k}(x_1, \ldots, x_n)$ such that

$$X \cap U = X_{(f_1, \ldots, f_{n-k})} \cap U,$$

and the $(n - k) \times n$ Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j} \right)$$

has rank $n - k$. Otherwise, we say that X *is singular at* x.

For example, in $\mathbb{A}^2$, the curve of equation

$$xy = 1$$

is nonsingular at $(1, 1)$. However, the curve of equation

$$xy = 0,$$

which is the union of two lines, is singular at $(0, 0)$. Using the Implicit Function Theorem, it can be shown that an affine variety X is nonsingular iff X is a complex manifold.