10.1 INTRODUCTION

The Fourier transform is a powerful tool in linear system analysis. It allows us to quantify the effects of digitizing systems, sampling spots, electronic amplifiers, convolution filters, noise, and display spots. Those who combine a theoretical knowledge of Fourier transform properties with a practical knowledge of their physical interpretation are well prepared to approach most image-processing problems. Usually, those who develop this combination of skills are students of electrical engineering and physics, and they do so in the course of their studies. For anyone who intends to use digital image processing seriously in their work, however, the time spent becoming familiar with the Fourier transform is well invested.

In a sense, the Fourier transform is like a second language for describing functions. Bilingual persons frequently find one language better than another for expressing certain ideas. Similarly, the image-processing analyst may move back and forth between the spatial domain and the frequency domain while processing through a problem.

When first learning a new language, one tends to think in his or her native tongue and mentally translate before speaking. After becoming fluent, however, one can think in either language. Similarly, once familiar with the Fourier transform, the analyst can think in either the spatial or the frequency domain, and this ability is quite useful.

In the first part of the chapter, we develop the properties of the Fourier transform using one-dimensional functions for simplicity of notation. Later we generalize the results to two dimensions. The convention in Part 2 of the text is first to consider one-dimensional functions as simple examples and then to extend the discussion to functions of two spatial variables as image-processing examples.

In our study of linear system analysis, we shall restrict our discussion to only one part of this well-developed field. For example, we use only the Fourier transform and not the Laplace transform or the Z-transform, because they are not required for our purposes. This restriction allows us to develop the techniques we need for the analysis of digital image-processing systems with a minimum of mathematical complexity.

One reason we do not require the generality of the Laplace transform, and other techniques from the field of linear system analysis, is that we are working with recorded data. This relieves us of the burden of dealing with physical realizability (causality) and its implications for the analysis.

Causality. Linear systems implemented with electronic hardware are referred to as causal because the input signal causes the output signal to occur. In particular, this means that if the input is zero for all negative time, then the output must likewise be zero for t < 0. While this is intuitively obvious, consider the constraint it places upon the impulse response of a linear system: If the input is an impulse at t = 0, the impulse response must be zero for all negative t. Thus, with physically realizable systems, the impulse response is always one sided. This means that it can be neither even nor odd, except in the trivial case. Such a condition considerably complicates the linear system analysis of physically realizable systems.

Working with recorded data leaves us not so constrained. Digitally implemented convolution can easily deal with even and odd functions, as well as those that are zero for negative time. Furthermore, in the spatial domain of image processing, the coordinate origin is arbitrary, and negative values of x and y have no special significance. Readers who find the mathematics in the following chapters burdensome should be thankful that we are working with recorded data and do not have to impose the causality condition upon the analysis.

10.1.1 The Continuous Fourier Transform

The Fourier transform of a one-dimensional function \( f(t) \) is defined as [1]

\[
\mathcal{F} \{ f(t) \} = F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} \, dt
\]

where \( j^2 = -1 \). The Fourier transform is a linear integral transformation that, in the general case, takes a complex function of n real variables into another complex function of n real variables. The inverse Fourier transform of \( F(s) \) is defined as

\[
\mathcal{F}^{-1} \{ F(s) \} = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} \, ds
\]

The only difference between the direct and inverse Fourier transformations is the sign of the exponent.

Fourier's integral theorem states that
of a Gaussian is also a Gaussian. This property makes the Gaussian function quite useful in analysis.

10.1.2 Existence of the Fourier Transform
Since the Fourier transform is an integral transformation, we must address the question of the existence of the integrals in Eqs. (1) and (2).

10.1.2.1 Transient Functions
Some functions go to zero for large positive and negative arguments rapidly enough that the integrals in Eqs. (1) and (2) exist. For our purposes, if the integral of the absolute value of a function exists, i.e., if
\[ \int_{-\infty}^{\infty} |f(t)| dt < \infty \]  
and the function either is continuous or has only finite discontinuities, then the Fourier transform of the function exists for all values of \( s \). We call these functions transient functions, since the useful ones characteristically die out at large \( t \).

In a sense, these are the only functions we shall ever process. Any digitized signal or image is necessarily truncated to finite duration and bandwidth. Thus, the transform exists for any function we shall ever be required to use. Nevertheless, it is convenient to be able to discuss other functions whose transforms do not exist in the strict sense.

10.1.2.2 Periodic and Constant Functions
Clearly, the Fourier transform does not exist for all values of \( s \) if \( f(t) = \cos(2\pi ft) \) or if \( f(t) = 1 \). However, the impulse \( \delta(t) \), introduced in Chapter 9, allows us to handle these cases conveniently. Consider the inverse transform of a pair of impulses
\[ f(t) = \mathcal{F}^{-1}\{ \delta(t - t_0) + \delta(t + t_0) \} = \int_{-\infty}^{\infty} [\delta(t - t_0) + \delta(t + t_0)] e^{2\pi i st} dt \]
which, by the shifting property of the impulse, is
\[ f(t) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{2\pi it} dt + \int_{-\infty}^{\infty} \delta(t + t_0) e^{2\pi it} dt = e^{2\pi i t_0} + e^{-2\pi i t_0} = 2 \cos(2\pi ft_0) \]
where we have used the Euler relation (Chapter 9, Eq. 7). Dividing by 2, we can write
\[ \mathcal{F}\{ \cos(2\pi ft_0) \} = \frac{1}{2}[\delta(t - t_0) + \delta(t + t_0)] \]
This means that the Fourier transform of a cosine of frequency \( f_0 \) is a pair of impulses located at \( s = \pm f_0 \) in the frequency domain. A similar development yields
\[ \mathcal{F}\{ \sin(2\pi ft_0) \} = \frac{j}{2}[\delta(t + t_0) - \delta(t - t_0)] \]
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If we let \( F(t) = 0 \) in Eq. (12), we can show that
\[
\mathcal{F}\{1\} = \delta(t)
\]

That is, the Fourier transform of a constant is an impulse at the origin.

We now have usable expressions for the Fourier transform of constant and sinusoidal functions. It is well known in the theory of Fourier series that any periodic function of frequency \( f \) can be expressed as a summation of sinusoids having frequencies \( nf \), where \( n \) takes on integer values. By the addition theorem (see Eq. (40)), this means that the Fourier transform of a periodic function is a series of equally spaced impulses in the frequency domain.

10.1.2.3 Random Functions

We lump nonconstant aperiodic functions of infinite extent whose absolute integral (Eq. (11)) does not exist into a class called random functions. In later chapters, we use these to model the output of a random process.

In most cases, we require only the autocorrelation function of a random function. This is given by

\[
R_x(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) f(t + T) dt
\]

and it exists for the functions that are of interest to us. The autocorrelation function is real and even, and its Fourier transform is the power spectrum of \( f(t) \), as is shown later.

If it becomes necessary to transform a random function, we can redefine the Fourier transform of Eq. (1) as

\[
F(\omega) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-j2\pi \omega t} dt
\]

and similarly for the inverse transform. We can then work with a class of functions for which these redefined transforms exist. In this book, however, we shall stay with the definitions set forth in Eqs. (1) and (2), since they are appropriate for bounded signals of finite duration. Any development carried out with this convention could be redone with the convention suggested by Eq. (16), thereby extending the result to random functions for which \( R_x(t) \) exists.

We conclude this discussion by taking the position that, for our purposes, the existence of the Fourier transform is not a major problem.

10.1.3 The Fourier Series Expansion

Suppose \( g(t) \) is a transient function that is zero outside the interval \([-T/2, T/2]\). This also can be considered to be one cycle of a periodic function. We can obtain a sequence of coefficients by making \( g(t) \) a discrete variable in Eq. (1) and integrating only over the interval, so that

\[
G_n = G(n\Delta t) = \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-j2\pi n t/T} dt
\]

where \( T \) is the period and \( \Delta t = 1/T \). This expansion represents \( g(t) \) by an infinite sequence of (complex-valued) coefficients, although, for many interesting functions, only finitely many of the coefficients are nonzero.

The inverse transform becomes

\[
g(t) = \sum_{n=-\infty}^{\infty} G(n\Delta t) e^{j2\pi n t/T} = \frac{1}{T} \sum_{n=-\infty}^{\infty} G_n e^{j2\pi n t/T}
\]

It reconstructs \( g(t) \) within the interval by adding together sinusoids of different frequencies. The amplitudes of these sinusoids are the coefficients \( G_n \).

The Fourier series expansion of the function \( f(t) \) is [1]

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n f t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f t)
\]

where

\[
a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(2\pi n f t) dt \quad \text{and} \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(2\pi n f t) dt
\]

It represents a periodic function of period \( T \) by two infinite sequences of real coefficients.

10.1.4 The Discrete Fourier Transform

If we discretize both time and frequency the Fourier transform of Eq. (19a) becomes

\[
G_n = G(n\Delta t) = \sum_{k=-N/2}^{N/2} g(k\Delta t) e^{j2\pi n k/T} = \frac{1}{N} \sum_{k=-N/2}^{N/2} G_k e^{j2\pi k n/T}
\]

where \( T = N\Delta t \). The inverse transform takes the form

\[
g_t = g(t) = \sum_{k=-N/2}^{N/2} G_k e^{j2\pi k t/T} = \frac{1}{T} \sum_{k=-N/2}^{N/2} G_k e^{j2\pi k t/T}
\]

Again, for many interesting functions, \( g(t) \), the coefficients \( G_k \) are nonzero only for relatively small \( n \).

If \( \{f_i\} \) is a sequence of length \( N \), such as that obtained by taking samples of a continuous function at equal intervals, then its discrete Fourier transform (DFT) is the sequence \( \{F_n\} \) given by

\[
F_n = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} f_l e^{-j2\pi l n/N}
\]

and the inverse DFT is

\[
f_l = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} F_n e^{j2\pi l n/N}
\]

where \( 0 \leq l, n \leq N - 1 \) are indices.
10.1.4.1 Relationship to the Continuous Transform

The similarity the DFT holds with Eqs. (1) and (2) and with Eqs. (20a) and (20b) suggests that the DFT might have many of the same properties as the integral transform. For the types of functions we work with in digital image processing, the differences are slight indeed. In fact, if \( f(t) \) is obtained by properly sampling a certain common type of continuous function, then the DFT can be shown to be a special case of the continuous Fourier transform [2].

Properly sampling these so-called bandlimited functions, and using the DFT to compute Fourier transforms are discussed in Chapters 12 and 13. Using the DFT to implement linear filtering is addressed in Chapter 16.

It is our good fortune that the DFT is so closely related to the continuous Fourier transform. As long as we abide by the sampling rules laid out in Chapter 12 we can view them as essentially equivalent. This flexibility affords us considerable latitude in the design process. It means, for example, that we can employ the continuous approach when formulating a solution to an image processing problem, and then implement that solution with the discrete approach.

10.1.5 The Fast Fourier Transform

When it is necessary to actually compute the Fourier transform of a sampled signal or image, we normally use the DFT. The number of multiplication and addition operations required to implement Eqs. (21) or (22) is clearly on the order of \( N^2 \), even after the required values of the complex exponential have been stored in a table. This makes the computation potentially burdensome.

Fortunately, there exists a class of algorithms that reduce the required number of operations to the order of \( N \log_2(N) \) [2-6]. These are called fast Fourier transform (FFT) algorithms. \( N \) must be factorable into a product of small integers. Highest efficiency and the simplest implementation result when \( N \) is a power of 2 (i.e., \( N = 2^p \) where \( p \) is an integer).

Notice that Eq. (21) can be written as the matrix product
\[
\begin{bmatrix}
F_0 \\
F_1 \\
\vdots \\
F_{N-1}
\end{bmatrix} =
\begin{bmatrix}
W_0,0 & \cdots & W_0,N-1 \\
\vdots & \ddots & \vdots \\
W_{N-1,0} & \cdots & W_{N-1,N-1}
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{N-1}
\end{bmatrix}
\]

or
\[
F = \mathcal{F} f
\]

where
\[
w_{k,l} = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi kl}{N}}
\]

Since the exponential function is periodic in the product of \( n \) and \( i \), there is considerable symmetry in the matrix \( \mathcal{F} \). The matrix can be factored into a product of \( N \) by \( N \) matrices that contain repeated values, including many zeros and ones [2]. If \( N = 2^p \), \( \mathcal{F} \) factors into \( p \) such matrices. The total number of operations required to implement \( p \) of those matrix products is substantially less than that required for Eq. (23).

10.2 PROPERTIES OF THE FOURIER TRANSFORM

10.2.1 Symmetry Properties

In the general case, a complex-valued function of a single real variable has a Fourier transform that is also a complex-valued function of a real variable. However, there are several restricted classes of functions that are of particular interest because of how their symmetry properties make them behave under the Fourier transformation.

10.2.1.1 Evenness and Oddness

A function \( f_1(t) \) is even if and only if
\[
f_1(t) = f_1(-t)
\]

and a function \( f_2(t) \) is odd if and only if
\[
f_2(t) = -f_2(-t)
\]
Sec. 10.2 Properties of the Fourier Transform

A function \( f(t) \) that is neither even nor odd can be broken into even and odd components given, respectively, by

\[
f_e(t) = \frac{1}{2} [f(t) + f(-t)]
\]

and

\[
f_o(t) = \frac{1}{2} [f(t) - f(-t)]
\]

where

\[
f(t) = f_e(t) + f_o(t)
\]

We now investigate the effect of evenness and oddness on the Fourier transform. Recall the Euler relation

\[ e^{j\theta} = \cos \theta + j\sin \theta \]

We can rewrite the Fourier transform [Eq. (1)] as

\[
F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st} dt = \int_{-\infty}^{\infty} f(t) \cos (2\pi st) dt - j\int_{-\infty}^{\infty} f(t) \sin (2\pi st) dt
\]

Expressing \( f(t) \) as a sum of even and odd components [Eq. (3)] produces

\[
F(s) = \int_{-\infty}^{\infty} f_e(t) \cos (2\pi st) dt + \int_{-\infty}^{\infty} f_o(t) \cos (2\pi st) dt
\]

\[- j\int_{-\infty}^{\infty} f_o(t) \sin (2\pi st) dt + j\int_{-\infty}^{\infty} f_e(t) \sin (2\pi st) dt
\]

Notice that the second and third terms are infinite integrals of the product of an even and an odd function. These terms evaluate to zero, and the Fourier transform reduces to

\[
F(s) = \int_{-\infty}^{\infty} f_e(t) \cos (2\pi st) dt - j\int_{-\infty}^{\infty} f_o(t) \sin (2\pi st) dt = F_e(s) \ast F_o(s)
\]

Now we can list the symmetry properties of the Fourier transform:

1. An even component function produces an even component function in the transform.
2. An odd component function produces an odd component function in the transform.
3. An odd component function introduces the coefficient \(-j\).
4. An even component function does not introduce a coefficient.

### 10.2.1.2 Real and Imaginary Components

We can use the preceding four rules to deduce the effect of the Fourier transformation on complex functions. If we express a general complex function as a sum of four components—an even and an odd real part, plus an even and an odd imaginary part—we can write the following four rules for the Fourier transformation:

1. The real even part produces a real even part.
2. The real odd part produces an imaginary odd part.
3. The imaginary even part produces an imaginary even part.
4. The imaginary odd part produces a real odd part.

Of particular interest is the case of input functions that are real, since we ordinarily use real functions to represent input images. Notice that a real function produces a transform that has an even real part and an odd imaginary part. This is referred to as a Hermite function, and it has the conjugate symmetry property

\[
F(s) = F^*(-s)
\]

where \( \ast \) denotes the complex conjugate.

Table 10.3 lists the full expansion of the symmetry properties of the Fourier transform. Notice that the inverse transformation [Eq. (2)] differs from the direct transformation [Eq. (1)] only in the sign of the odd component. This tells us that the forward and inverse transforms of an even function are the same.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( \mathcal{F}{f(t)} = F(s) )</th>
<th>( g(t) )</th>
<th>( \mathcal{F}{g(t)} = G(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Even</td>
<td>Odd</td>
<td>Odd</td>
</tr>
<tr>
<td>Real and even</td>
<td>Real and even</td>
<td>Imaginary and odd</td>
<td>Imaginary and odd</td>
</tr>
<tr>
<td>Real and odd</td>
<td>Imaginary and even</td>
<td>Complex and even</td>
<td>Complex and even</td>
</tr>
<tr>
<td>Imaginary and even</td>
<td>Complex and even</td>
<td>Real</td>
<td>Real</td>
</tr>
<tr>
<td>Complex and even</td>
<td>Complex and odd</td>
<td>Imaginary</td>
<td>Imaginary</td>
</tr>
<tr>
<td>Complex and odd</td>
<td>Real</td>
<td>Anti-Hermite</td>
<td>Real</td>
</tr>
<tr>
<td>Real</td>
<td>Real odd, plus imaginary and odd</td>
<td>Real and odd, plus imaginary and even</td>
<td>Real and odd, plus imaginary and even</td>
</tr>
</tbody>
</table>

#### 10.2.2 The Addition Theorem

Suppose we have two Fourier transform pairs

\[
\mathcal{F}\{f(t)\} = F(s)
\]

and

\[
\mathcal{F}\{g(t)\} = G(s)
\]

If the two time functions are added, the Fourier transform of their sum is

\[
\mathcal{F}\{f(t) + g(t)\} = \int_{-\infty}^{\infty} [f(t) + g(t)] e^{-j2\pi st} dt
\]

This may be rearranged to yield

\[
\mathcal{F}\{f(t) + g(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st} dt + \int_{-\infty}^{\infty} g(t)e^{-j2\pi st} dt = F(s) + G(s)
\]
Sec. 10.2  Properties of the Fourier Transform

Thus, addition in the time or spatial domain corresponds to addition in the frequency domain, as illustrated in Figure 10-1. This fits well with the concept of linearity in a system. It follows from the addition theorem that

\[ \mathcal{F} \{ cf(t) \} = cF(f) \]  \hspace{1cm} (41)

where \( c \) is a rational constant. We take it as an axiom that Eq. (41) holds for any constant. 

\[ f(\theta) \]

\[ p(\theta) \]

\[ a(\theta) \]

\[ G(f) \]

\[ f(t) + g(t) \]

\[ a(f) + G(f) \]

\[ F(f + a) \]

\[ f(\theta) * g(\theta) \]

\[ F(f) \ast G(f) \]

Figure 10-1  The addition theorem

10.2.3  The Shift Theorem

The shift theorem describes the effect that moving the origin of (shifting) a function has upon its transform. Using the function \( f(t) \) as before, we can write

\[ \mathcal{F} \{ f(t - a) \} = \int_{-\infty}^{\infty} f(t - a)e^{-j2\pi ft}dt \]  \hspace{1cm} (42)

where \( a \) is the amount of shift. Multiplying the right-hand side of the equation by

\[ e^{j2\pi ft}e^{-j2\pi ft} = 1 \]

provides

\[ \mathcal{F} \{ f(t - a) \} = \int_{-\infty}^{\infty} f(t - a)e^{-j2\pi ft}dt \]  \hspace{1cm} (43)

Next, we make the variable substitution

\[ u = t - a \quad du = dt \]

and move the second exponential outside the integral, leaving

\[ \mathcal{F} \{ f(t - a) \} = e^{-j2\pi fu} \mathcal{F} \{ f(u + a) \} \]  \hspace{1cm} (44)

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\[ \mathcal{F} \{ f(t - a) \} = e^{-j2\pi fu} \int_{-\infty}^{\infty} f(u)e^{j2\pi fu}du = e^{-j2\pi fu}F(s) \]  \hspace{1cm} (46)

Thus, shifting a function introduces a complex exponential coefficient into its Fourier transform. Notice that if \( a = 0 \), this coefficient is unity. The complex coefficient

\[ e^{-j2\pi fu} = \cos(2\pi fu) - j\sin(2\pi fu) \]  \hspace{1cm} (47)

has unit magnitude and revolves in the complex plane with increasing \( s \). This means that shifting a function does not change the amplitude (modulus) of its Fourier transform, but does alter the distribution of energy between its real and imaginary parts. The result is a phase shift proportional to both frequency and \( a \), the amount of shift.

10.2.4  The Convolution Theorem

Perhaps the most important theorem for linear system analysis is the convolution theorem. We can express the Fourier transform of the convolution of the functions given in Eqs. (37) and (38) as

\[ \mathcal{F} \{ f(t) * g(t) \} = \int_{-\infty}^{\infty} f(u)g(t - u)du e^{-j2\pi ft}dt \]  \hspace{1cm} (48)

which, after rearrangement, becomes

\[ \mathcal{F} \{ f(t) * g(t) \} = \int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty} g(t - u)e^{-j2\pi ft}dt du \]  \hspace{1cm} (49)

By the shift theorem, we can write

\[ \mathcal{F} \{ f(t + a) \} = \int_{-\infty}^{\infty} f(u)e^{-j2\pi fu}G(u + a)du = G(s)\int_{-\infty}^{\infty} f(u)e^{-j2\pi fu}du \]  \hspace{1cm} (50)

This means that

\[ \mathcal{F} \{ f(t) * g(t) \} = F(s)G(s) \]  \hspace{1cm} (51)

and convolution in one domain corresponds to multiplication in the other domain. It follows that

\[ \mathcal{F}^{-1} \{ F(s)G(s) \} = f(t + a) \ast g(t) \]  \hspace{1cm} (52)

The convolution theorem points out a major benefit of the Fourier transform: Rather than performing convolution in one domain, which is complicated to visualize and expensive to implement, we can perform multiplication in the other domain for the same effect.

We can use the convolution theorem to derive the Fourier transform of the impulse. Recall that

\[ f(t) \ast \delta(t) = f(t) \]  \hspace{1cm} (53)

that is, the impulse is the identity under convolution. By the convolution theorem,

\[ f(t) \mathcal{F} \{ \delta(t) \} = F(s) \]  \hspace{1cm} (54)
Sec. 10.2 Properties of the Fourier Transform

Since this is true for any \( f(t) \), we can choose one such that \( F(s) \) has no zeros—for example, he Gaussian. Then we can divide by \( F(s) \) to show that

\[
\mathcal{F}\{ h(t) \} = 1
\]

proving that the Fourier transform of the impulse is unity.

10.2.5 The Similarity Theorem

The similarity theorem describes the effect that a change in scale of the abscissa has on the Fourier transform of a function.

Changing the abscissa's scale broadens or narrows a function. Thus, we can stretch or compress the function given in Eq. (37) by placing a coefficient in its argument. Its Fourier transform then becomes

\[
\mathcal{F}\{ f(at) \} = \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{i2\pi ut/a} du
\]

Multiplying both the integral and the exponent by \( a/d \) produces

\[
\mathcal{F}\{ f(at) \} = \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{i2\pi ut/a} du
\]

We now make the variable substitution

\[
x = at, \quad dx = a dt
\]

and write

\[
\mathcal{F}\{ f(at) \} = \frac{1}{|a|} \int_{-\infty}^{\infty} f(u) e^{i2\pi ut/a} du
\]

which we recognize as

\[
\mathcal{F}\{ f(at) \} = \frac{1}{|a|} \mathcal{F}\{ \frac{f(u)}{a} \}
\]

If the coefficient \( a \) is greater than unity, it contracts the function \( f(t) \) horizontally, which, by Eq. (60), reduces the amplitude of the Fourier transform and expands it horizontally by the factor \( a \). If \( a \) is less than unity, it has the opposite effect. This is illustrated in Figure 10.2. The similarity theorem implies that a narrow function has a broad Fourier transform and vice versa.

We can use the similarity theorem to derive a general expression for the Fourier transform of a Gaussian. Recall from Eqs. (5) and (12) that the Fourier transform of a Gaussian is also a Gaussian:

\[
\mathcal{F}\{ e^{-\alpha t^2} \} = e^{-\alpha \omega^2}
\]

By the similarity theorem:

\[
\mathcal{F}\{ e^{-|\alpha| \omega^2} \} = \frac{1}{|\alpha|} e^{-\alpha \omega^2}
\]

Figure 10.2 The similarity theorem

We now let

\[
e^{-\alpha \omega^2} = e^{-\alpha t^2} = e^{-\alpha \omega^2}
\]

and solve for

\[
\alpha = \frac{1}{\sqrt{2\pi \sigma^2}}
\]

Now the transform is given by

\[
\mathcal{F}\{ e^{-\alpha \omega^2} \} = \frac{1}{\sqrt{2\pi \alpha^2}} e^{-\alpha \omega^2}
\]

but since it, too, is a Gaussian, we can define a standard deviation \( \sigma \) such that

\[
e^{-\alpha \omega^2} = e^{-\alpha t^2}
\]

This means that

\[
2\pi \sigma^2 \alpha^2 = \frac{\lambda}{\alpha}
\]

or

\[
\alpha = \frac{1}{\sqrt{2\pi \sigma^2}}
\]

So the Fourier transform of a Gaussian of arbitrary standard deviation \( \sigma \) is

\[
\mathcal{F}\{ e^{-\alpha \omega^2} \} = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\alpha \omega^2} \quad \alpha = \frac{1}{\sqrt{2\pi \sigma^2}}
\]
Thus, the Fourier transform of a unit-amplitude Gaussian with standard deviation $\sigma$ is another Gaussian with amplitude $\sqrt{2\pi}\sigma$ and standard deviation $1/(2\pi\sigma)$.

We can use the similarity theorem to illustrate again that the transform of the impulse is constant. Suppose that

$$f(t) = e^{-\alpha t^2}$$

and its transform is

$$F(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/(2\pi\sigma^2)}$$

If we let $\alpha$ approach infinity, $f(t)$ narrows and grows in amplitude to approach an impulse, while $F(x)$ expands to approach constant unit amplitude. Thus, in the limiting case, the shrinking Gaussian approaches an impulse, and its expanding Gaussian transform approaches unity.

### 10.3.2 Rayleigh’s Theorem

An important class of functions is those that are nonzero only over a finite portion of their domain. For such functions, we can discuss the total energy content. The energy of a function is defined as

$$\text{energy} = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

provided that the integral exists. For transient functions, the integral in Eq. (72) exists, and the energy is a convenient parameter reflecting the total "size" of the function. Rayleigh’s theorem states that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(x)|^2 dx$$

which means that the transform has the same energy as the original function.

The proof of Rayleigh’s theorem is as follows. First we write

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t)f^*(t)dt = \int_{-\infty}^{\infty} f(t)f^*(t)e^{-2\pi i u t}dt$$

that is, the second equality holds for $u = 0$. Again, we use the superscript asterisk to indicate the complex conjugate, since $f(t)$ is, in general, complex. We recognize Eq. (74) as the inverse Fourier transform of a product of two functions evaluated at the frequency $u = 0$. Since

$$\mathcal{F}^{-1}\{f(t)f^*(t)\} = \mathcal{F}^{-1}\{f(t)\} \ast \mathcal{F}^{-1}\{f(t)\} = \delta(u)$$

we can write the convolution integral as

$$\mathcal{F}^{-1}\{f(t)f^*(t)\} = \int_{-\infty}^{\infty} F(x)F(x)dx$$

Substituting $u = 0$ produces

$$\mathcal{F}^{-1}\{f(t)f^*(t)\} = \int_{-\infty}^{\infty} F(x)F(x)dx$$

which proves Eq. (73) and states that the energy is the same in both domains. If $f(t)$ is real and even, then $F(x)$ is also real and even, and

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{\infty} F(x)dx$$

Notice how Rayleigh’s theorem agrees with the similarity theorem: If we narrow a function at constant amplitude, we clearly reduce its energy. The similarity theorem states that narrowing a function broadens its transform, but also reduces its amplitude, keeping the energy equal in both domains.

### 10.3 Linear Systems and the Fourier Transform

In this section, we examine the important role the Fourier transform plays in linear system analysis.

#### 10.3.1 Linear System Terminology

Figure 10–3 shows, in both domains, the terminology commonly used for a linear system. In general, the Fourier transform of a signal is called the spectrum of that signal, and the inverse Fourier transform of a spectrum is a signal. Similarly, the impulse response and the transfer function form a Fourier transform pair.

#### 10.3.2 Linear System Identification

Frequently, the impulse response and transfer function of a system are unknown and must be determined. This process is called system identification. For the linear system shown in Figure 10–3, the convolution theorem implies that

$$H(s) = \frac{H(s)}{F(s)}$$

We can now write

$$G(s) = \frac{H(s)}{F(s)}$$

where

- $f(t)$ is the input signal
- $F(s)$ is the spectrum of the input signal
- $g(t)$ is the impulse response
- $G(s)$ is the transfer function
- $m(t)$ is the output signal
- $H(s)$ is the spectrum of the output signal

![Figure 10–3 Linear system terminology](image-url)
Sec. 10.3  Linear Systems and the Fourier Transform

and therefore,

\[ g(t) = \mathcal{F}^{-1} \left[ \mathcal{F} \left\{ h(t) \right\} \right] \]

(81)

This means that we can input a known \( f(t) \), measure \( h(t) \), and compute \( g(t) \) by numerical integration. For instance, suppose \( f(t) \) is an impulse. Then \( h(t) \) is merely the impulse response, and no further action is necessary to identify the system.

As a more interesting example, assume that

\[ f(t) = H(t) \]

(82)

\( s \) the input, and

\[ h(t) = H(t) \]

(83)

\( s \) measured at the output, as shown in Figure 10-4. Now

Figure 10-4  System identification, example 1

\[ g(t) = \mathcal{F}^{-1} \left[ \frac{\sin^2(\pi s)}{s^2} \right] = \Pi(t) \]

(84)

is the impulse response.

As a second example, consider Figure 10-5. Suppose we choose as an input

Figure 10-5  System identification, example 2
Sec. 10.3 Linear Systems and the Fourier Transform

\[ f(t) = u(t) - \frac{1}{2} = \begin{cases} \frac{1}{2}, & t < 0 \\ 0, & t = 0 \\ \frac{1}{2}, & t > 0 \end{cases} \] (85)

which is an edge function having the spectrum

\[ F(s) = \frac{j}{2\pi} \] (86)

If the system's response is given by

\[ h(t) = \begin{cases} \frac{1}{2}, & t < 0 \\ 0, & -1 \leq t \leq 1 \\ \frac{1}{2}, & t > 1 \end{cases} \] (87)

which has the spectrum

\[ H(s) = \frac{\sin(\pi s)}{2(\pi s)^2} \] (88)

we can write

\[ G(s) = \frac{H(s)}{F(s)} = \frac{\sin(\pi s)}{\pi s} \] (89)

which implies that the impulse response is

\[ g(t) = \Pi(t) \] (90)

In the preceding examples, the system output was expressed analytically and the problem solved directly. In the usual case, however, the process goes more like this: The output is digitized, both input and output are transformed by numerical integration, the ratio in Eq. (80) is computed directly, and the inverse Fourier transformation of Eq. (81) is performed by numerical integration. The fast Fourier transform, a computationally efficient algorithm for computing the Fourier transform, is most commonly used [3–10].

Notice that it is prudent to choose an input function whose spectrum does not have zeros. In the second example, we violate this constraint, but are fortunate enough to encounter an impulse response that also has zeros at those points in the frequency domain. If \( F(s) \) has zero-crossings, \( H(s) \) will as well, and \( G(s) \) can be interpolated from surrounding values before the inverse transformation is performed numerically.

10.3.3 Sinusoidal Decomposition

The Fourier transform is a linear integral transform that uses the imaginary exponential as its kernel function. As shown in Eq. (33), the Fourier transform can be expressed as a sum of two transforms using the sine and cosine functions as kernels. Thus, a should come as no surprise that sine and cosine functions exhibit specialized behavior under Fourier transformation.

The following exercise yields insight into the relationship between the impulse response and the transfer function of a linear system. Consider again the linear system shown in Figure 10-3, and assume, for graphical convenience, that \( f(t) \) and \( g(t) \) are real and even. In Figure 10-6, the input and the impulse response are graphed in both domains.

For the input spectrum, let us divide the \( s \)-axis into small equal intervals \( \Delta s \) and divide \( F(s) \) into narrow strips \( \Delta s \) wide. If \( \Delta s \) is sufficiently small, \( F(s) \) is well approximated by a sum of rectangular pulses, as shown in Figure 10-7. Note that an approximation to \( F(s) \) is given by an infinite summation of such pulse pairs:

![Figure 10-6 Linear system example](image)

![Figure 10-7 Sinusoidal decomposition](image)
Sec. 10.3 Linear Systems and the Fourier Transform

\[ F(s) = \sum_{\alpha} F(\alpha s) \left[ \int \frac{x(s - \alpha s)}{\Delta s} + \int \frac{x(s + \alpha s)}{\Delta s} \right] \]  

(91)

Consider a particular pair of pulses, namely, those situated at \( s = \pm \Delta s \) and having amplitude \( F(s) \Delta s \) and area \( F(s) \Delta s \). As \( \Delta s \) approaches zero, the pulse pair approaches an even impulse pair at \( s = \pm \Delta s \), with infinitesimal strength \( F(s) \Delta s \). The inverse transform of the even pulse pair approaches

\[ 2F(s)\Delta s \cos(2\pi s) \]  

(92)

Since \( F(s) \) approaches a sum of even pulse pairs [Eq. (91)], \( f(t) \) approaches a sum of cosines of the form of Eq. (92). This means that any even function can be decomposed into a sum of infinitely many cosines of infinitesimal amplitude.

Since the output spectrum is the product of the input spectrum and the transfer function, the output signal \( h(t) \) can be expressed as a sum of cosines of the form

\[ 2G(s)F(s)\Delta s \cos(2\pi s) \]  

(93)

We may now view the action of a linear filter as follows. The input signal \( f(t) \) is first decomposed into a sum of cosines of all different frequencies. The amplitudes of the individual cosines are uniquely determined by \( F(s) \), which is in turn the (unique) Fourier transform of \( f(t) \).

Inside the linear system, each cosine of frequency \( s_0 \) is multiplied by \( G(s_0) \), the amplitude of the transfer function evaluated at its frequency. Finally, all the cosines of modified amplitude are summed at the output of the filter to form the output signal \( h(t) \).

Notice that this interpretation is consistent with two previously discussed properties of linear systems: (1) A sinusoidal input always produces a sinusoidal output at the same frequency, and (2) the transfer function at frequency \( s \) is the factor by which the amplitude of an input sinusoid of frequency \( s \) is multiplied.

If we had made \( f(t) \) odd, \( F(s) \) would have been imaginary and odd, and \( f(t) \) would then decompose into sine functions. The remainder of the process would be identical, except that the output, the sine functions of modified amplitude would be summed to produce the odd output signal \( h(t) \).

Similarly, if \( f(t) \) were neither even nor odd, it could first be decomposed into even and odd component functions, each of which would then be decomposed as before into cosines and sines, respectively. The modified cosines and sines again would be summed at the output to produce the output signal \( h(t) \).

The foregoing discussion assumes that the transfer function is real and even. Suppose instead that the input is real and even. But the impulse response \( g(t) \) is real and odd. This makes the transfer function imaginary. When the incoming even pulse pairs are multiplied by the imaginary odd transfer function, they are converted into imaginary odd impulse pairs. The process converts the incoming cosines into output sines. The output then becomes a summation of sine functions, and \( g(t) \) is odd.

The preceding indicates that convolving an even input function with an odd impulse response produces an odd function. From the graphical interpretation of the convolution integral, one can satisfy oneself that this is correct.

Consider now the case where the input function is a cosine, viz.,

\[ f(t) = \cos(2\pi f_0) \]  

(94)

and the impulse response is real, consisting of even and odd components, i.e.,

\[ g(t) = g_e(t) + jg_0(t) \]  

(95)

The transfer function

\[ G(s) = G_e(s) + jG_0(s) \]  

(96)

is Hermite, which means that

\[ G(f_0) = G_e(f_0) + jG_0(f_0) \]  

(97)

and

\[ G(-f_0) = G^*(f_0) = G_c(f_0) - jG_0(f_0) \]  

(98)

Recall that the spectrum of the cosine is

\[ F(s) = \frac{1}{2} [\delta(s - f_0) + \delta(s + f_0)] \]  

(99)

We can now write the output spectrum as

\[ H(s) = \frac{1}{2} G_e(f_0)[\delta(s - f_0) + \delta(s + f_0)] + \frac{1}{2} jG_0(f_0)[\delta(s - f_0) - \delta(s + f_0)] \]  

(100)

which means that the output signal is

\[ h(t) = G_e(f_0)\cos(2\pi f_0 t) + G_0(f_0)\sin(2\pi f_0 t) \]  

(101)

This can be written as

\[ h(t) = \frac{A}{2} \cos(2\pi f_0 t + \phi) \]  

(102)

where

\[ A = \sqrt{G_e(f_0)^2 + G_0(f_0)^2} \quad \text{and} \quad \phi = \arctan \frac{G_0(f_0)}{G_e(f_0)} \]  

(103)

This is an expected result in view of the property that a linear system can change the amplitude and phase of a sinusoidal input, but cannot change its frequency or functional form.

The foregoing exercise illustrates the relationship between the even and odd components of a real impulse response and the real and imaginary components of the transfer function. It shows how the component of the impulse response introduces an imaginary odd component into the transfer function. This produces a sine component output from a cosine component of the input and reflects itself in phase shift at the output. Finally, it illustrates that the amplitude of the output depends on the root-mean-square amplitude (modulus) of the complex transfer function.

Notice that we now have two equivalent ways of viewing the operation of a linear system: (1) We may visualize convolution, with functions being reflected, shifted, multiplied, and integrated, or (2) we may visualize sinusoidal decomposition followed by multiplication and resumation. We also understand the restrictions that evenness and oddness in one domain place on the functions in the other domain. Having these two options available affords us a very useful flexibility when approaching a problem with linear system analysis. It also illustrates the bilingual analogy mentioned at the beginning of this chapter.
10.3.4 Negative Frequency

Persons having prior experience with radio transmission or the use of a waveform analyzer or spectrum analyzer are sometimes uncomfortable with the concept of frequencies less than zero. Waveform and spectrum analyzers incorporate narrow bandpass filters that allow energy to pass only in a narrow range about certain sinusoidal frequencies. These filters act to select, out of a signal, the sinusoidal component at a particular frequency.

One can derive the spectrum of an electrical signal by tuning the narrow-band filter across the (positive) frequency range and plotting the amplitude of the output. Persons experienced in using this type of equipment may be unfamiliar with the concept of negative frequency.

Recall that the Fourier transform of the cosine is an even impulse pair and the transform of the sine is an imaginary odd impulse pair. Since the cosine is an even function, it must have an even spectrum, and similarly for oddness and the sine function.

For any real function, the spectrum is Hermitian, and the left half is merely a complex conjugate reflection of the right half. For real functions, then, we are using a double-sided mathematical technique somewhat redundantly.

Since the left half of the spectrum is redundant for real functions, it could be ignored, as it implicitly is ignored in the use of a spectrum analyzer. However, we are using a somewhat more general mathematical approach to model physical processes, and the analysis is much simpler if we retain the left half of the functions.

Throughout Part 2 of the text, we graph double-sided spectra, although spectra are often plotted elsewhere only for positive frequency. We should keep in mind that, as long as we are using double-sided mathematics to model the operation of linear systems, the left half of the function, redundant though it may be, is a part of the analysis.

10.4 THE FOURIER TRANSFORM IN TWO DIMENSIONS

So far, we have considered the Fourier transform of one-dimensional functions of time. In digital image processing, and in the analysis of optical systems, the inputs and outputs are commonly two dimensional and, in some cases, higher dimensional. Our investment in the one-dimensional Fourier transform will not prove to be wasted effort, however, since the transform generalizes easily to higher dimensions.

10.4.1 Definition

For functions of two dimensions, the direct and inverse Fourier transforms are respectively defined as

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(u x + v y)} \, dx \, dy \]  \hspace{1cm} (104)

and

\[ f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(u x + v y)} \, du \, dv \]  \hspace{1cm} (105)

where \( f(x, y) \) is an image and \( F(u, v) \) is its spectrum. \( F(u, v) \) is, in general, a complex-valued function of two real frequency variables \( u \) and \( v \). The variable \( u \) corresponds to frequency along the \( x \)-axis, and similarly for \( v \) and the \( y \)-axis.

Figure 10-8 shows an image and its two-dimensional amplitude spectrum. Gray level is scaled to represent the magnitude (square root of the sum of the squares of the real and imaginary parts) at each point \( u, v \) in two-dimensional frequency space. The origin is located at the center of the transform image. Periodic noise in the image produces the spikes in the transform.

10.4.2 The Two-Dimensional DFT

If \( g(i, k) \) is an \( N \times N \) array, such as that obtained by sampling a continuous function of two dimensions at equal intervals on a rectangular grid, then its two-dimensional discrete Fourier transform (DFT) is the array given by

\[ G(m, n) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} g(i, k) e^{-j2\pi(\frac{m i}{N} + \frac{n k}{N})} \]  \hspace{1cm} (106)

and the inverse DFT is

\[ g(i, k) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} G(m, n) e^{j2\pi(\frac{m i}{N} + \frac{n k}{N})} \]  \hspace{1cm} (107)

As in one dimension, the DFT is quite similar to the continuous Fourier transform. And as before, the two-dimensional DFT of a bandlimited function sampled on a rectangular grid is a special case of the continuous Fourier transform.

Separability. The exponential in Eq. (106) can be factored, allowing us to write the transformation as
Sec. 10.4   The Fourier Transform in Two Dimensions

\[ G(m, n) = \frac{1}{N^2} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x', y') e^{-j2\pi \frac{m x'}{N}} e^{-j2\pi \frac{n y'}{N}} \]  

(108)

by removing the transformation into horizontal and vertical operations. Here, the term in brackets represents one-dimensional DFTs computed on the rows of the image. The outer summation then performs a columnwise one-dimensional discrete Fourier transforms on the resulting array. Efficient implementations of the DFT in two dimensions use this approach along with the one-dimensional FFT. The inverse DFT is likewise separable.

10.4.3 Matrix Formulation

In matrix notation, the DFT can be written as

\[ G = F g F^T \]  

(109)

where

\[ F = \left[ \frac{1}{\sqrt{N}} e^{-j2\pi \frac{m n}{N}} \right] \]  

(110)

is an \( N \times N \) unitary matrix, that is, its inverse is the transpose of its complex conjugate. To invert a unitary matrix, one simply interchanges rows and columns and reverses the sign of the imaginary part of each element. Since \( F \) is also symmetric, the transposition is trivial.

Notice that row-stacking to form a column vector and the use of a large block-circulant matrix, as was required for two-dimensional convolution (Sec. 9.3.4), is not necessary for computing the two-dimensional DFT. This is because the kernel function is separable into rowwise and columnwise operations, and \( F \) is unitary.

10.4.4 Properties of the Two-Dimensional Fourier Transform

The theorems of the two-dimensional Fourier transform are summarized in Table 10–3. Notice that the generalization from one dimension to two is quite direct.

The two-dimensional Fourier transform has several properties that have no one-dimensional counterpart. One is the property that if a two-dimensional image factors into a product of one-dimensional components, the same is true for the two-dimensional spectrum of the image. Another is the rotation property, which proves valuable in computerized axial tomography (CAT) scanners, discussed in Chapter 22.

The Laplacian is an omnidirectional second-derivative operator often used for edge detection and edge enhancement. Notice that using the Laplacian on a function multiplies its spectrum by \( \delta_x^2 + \delta_y^2 \). For the convolution theorem, then, the Laplacian corresponds to a linear system with a transfer function that increases as the square of frequency.

10.4.4.1 Separability

Suppose that

\[ f(x, y) = f_x(x) f_y(y) \]  

(111)

Then

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_x(x) f_y(y) e^{-j2\pi x u} e^{-j2\pi y v} dx \, dy \]  

(112)

can be rearranged to yield

\[ F(u, v) = \int_{-\infty}^{\infty} f_x(x) e^{-j2\pi u x} dx \int_{-\infty}^{\infty} f_y(y) e^{-j2\pi v y} dy = F_x(u) F_y(v) \]  

(113)

Thus, if a two-dimensional image factors into one-dimensional components, its spectrum does as well.

Consider as an example the elliptical two-dimensional Gaussian

\[ e^{-x^2/(2\sigma_x^2)} e^{-y^2/(2\sigma_y^2)} = e^{-x^2/2\sigma_x^2} e^{-y^2/2\sigma_y^2} \]  

(114)

which factors into the product of two one-dimensional Gaussians. If the standard deviations of the two factors are equal, we have

\[ e^{-x^2 + y^2/2\sigma^2} = e^{-x^2/2\sigma^2} e^{-y^2/2\sigma^2} \]  

(115)

which is the circular Gaussian. This function is extremely useful in the analysis of optical systems because it has circular symmetry and yet can be factored into one-dimensional components.
Sec. 10.4 The Fourier Transform in Two Dimensions

10.4.4.2 Similarity

The similarity theorem may be generalized to the case of two-dimensional transforms. We can write

$$\mathcal{F}\{f(a_1x + b_1y, a_2x + b_2y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(a_1x + b_1y, a_2x + b_2y)e^{-2\pi j(ax + by)} dx \ dy \quad (116)$$

We make the substitutions

$$w = a_1x + b_1y \quad z = a_2x + b_2y \quad (117)$$

in which case

$$x = A_1w + B_1z \quad y = A_2w + B_2z \quad (118)$$

dx = A_1dw + B_1dz \quad dy = A_2dw + B_2dz \quad$$

where

$$A_1 = \frac{b_2}{a_1b_2 - a_2b_1} \quad B_1 = \frac{-b_1}{a_1b_2 - a_2b_1} \quad (119)$$

$$A_2 = \frac{-a_2}{a_1b_2 - a_2b_1} \quad B_2 = \frac{a_1}{a_1b_2 - a_2b_1}$$

Then the Fourier transform becomes

$$\mathcal{F}\{f(a_1x + b_1y, a_2x + b_2y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(w, z)e^{2\pi j(z\theta + w\phi)} dw \ dz \ \mathcal{F}\{A_1B_2 + A_2B_1\} \quad (120)$$

10.4.4.3 Rotation

From the two-dimensional similarity theorem, it follows that a rotation of $f(x, y)$ through an angle $\theta$ also rotates the spectrum of $f(x, y)$ by the same amount. We let

$$a_1 = \cos \theta \quad b_1 = \sin \theta \quad a_2 = -\sin \theta \quad b_2 = \cos \theta \quad (121)$$

so that

$$A_1 = \cos \theta \quad A_2 = \sin \theta \quad B_1 = -\sin \theta \quad B_2 = \cos \theta \quad (122)$$

and

$$\mathcal{F}\{f(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)\} = \mathcal{F}\{f(x, y)\}e^{-2\pi j(x\cos \theta + y\sin \theta)} \quad (123)$$

10.4.4.4 Projection

Suppose we collapse a two-dimensional function $f(x, y)$ into a one-dimensional function by projection onto the $x$-axis to form

$$p(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (124)$$

Then the (one-dimensional) Fourier transform of $p(x)$ is

$$P(u) = \int_{-\infty}^{\infty} F(u, 0) du \quad (125)$$

But $P(u)$ can be written as

$$P(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-2\pi j(xu + y\theta)} dx \ dy = F(u, 0) \quad (126)$$

so the transform of the projection of $f(x, y)$ onto the $x$-axis is $F(u, 0)$ evaluated along the $u$-axis. This combines with the rotation property to imply that the one-dimensional Fourier transform of $f(x, y)$ projected onto a line at an angle $\theta$ with the $x$-axis is just $F(u, 0)$ evaluated along a line at an angle $\theta$ with the $u$-axis (Figure 10-9). The projection property forms the basis for system identification by line spread functions (Chapter 16) and for computerized axial tomography (Chapter 22).

![Figure 10-9](image-url) The projection property of the two-dimensional Fourier transform

10.4.5 Circular Symmetry and the Huygens Transform

Many important two-dimensional functions exhibit the property of circular symmetry. This means that the function can be expressed as a profile function of a single radial variable

$$f(x, y) = f(r) \quad (127)$$

where

$$r^2 = x^2 + y^2 \quad (128)$$

We now investigate the effect that circular symmetry has upon the two-dimensional Fourier transform. We can write the Fourier transform of $f(x, y)$ as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-2\pi j(x\cos \theta + y\sin \theta)} dx \ dy = \int_{0}^{\infty} \int_{0}^{2\pi} f(r) e^{-2\pi j r \cos \theta} r dr \ d\theta \quad (129)$$
where we have converted the integration from rectangular to annular and made the variable substitution

$$z = re^{i\theta} \quad \text{and} \quad u + jv = qe^{i\phi}$$

We can now rearrange Eq. (129), dropping $\phi$ because the integral is taken over a full cycle of the cosine, to yield

$$\mathbf{\mathcal{F}} \{ f(x, y) \} = \int_0^{2\pi} f(x) \left( \int_0^\infty e^{-i(2\pi \nu r \cos \theta)} r \, dr \right) \, d\theta$$

(131)

Now consider the integral in brackets, and recall the definition of the zero-order Bessel function of the first kind:

$$J_0(z) = \frac{1}{2\pi i} \int_0^{2\pi} e^{iz \cos \theta} \, d\theta$$

(132)

Recognizing Eq. (132) in Eq. (131) allows us to write

$$\mathbf{\mathcal{F}} \{ f(x, y) \} = 2\pi \int_0^{2\pi} f(x) J_0(2\pi \nu r) \, r \, dr$$

(133)

Notice that the Fourier transform of a circularly symmetric function is a function only of a single radial frequency variable $\nu$. This means that

$$F(u, v) = F(q)$$

(134)

where

$$q^2 = u^2 + v^2$$

(135)

### 10.4.5.1 The Hankel Transform

For circularly symmetric functions, the direct transform is

$$F(q) = 2\pi \int_0^{\infty} f(r) J_0(2\pi \nu r) \, r \, dr$$

(136)

and the inverse transform is

$$f(r) = 2\pi \int_0^{\infty} F(q) J_0(2\pi \nu r) \, \nu \, d\nu$$

(137)

These equations define a special case of the two-dimensional Fourier transform that is called the Hankel transform of zero order. This transform is a one-dimensional linear integral transform similar to the Fourier transform, except that the kernel is a Bessel function. Hence, two-dimensional functions with circular symmetry may be treated as one-dimensional functions of a single radial variable if the Hankel transform is substituted for the Fourier transform.

Hankel transforms of some familiar functions are listed in Table 10-4. Table 10-5 illustrates the theorems of the Hankel transform.

### Table 10-4 HANKEL TRANSFORMS OF CERTAIN FUNCTIONS

<table>
<thead>
<tr>
<th>Function</th>
<th>$f(r)$</th>
<th>$F(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reciprocal</td>
<td>$\frac{1}{r}$</td>
<td>$\frac{1}{q}$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$e^{-\alpha^2 r}$</td>
<td>$e^{-\nu^2}$</td>
</tr>
<tr>
<td>Impulse</td>
<td>$\delta(r)$</td>
<td>$1$</td>
</tr>
<tr>
<td>Rectangular pulse</td>
<td>$\frac{1}{2\nu}$</td>
<td>$\frac{a}{\sqrt{2\pi a^2 q}}$</td>
</tr>
<tr>
<td>Triangular pulse</td>
<td>$\frac{1}{2\nu}$</td>
<td>$\frac{2\pi}{a^2} \int_0^a f(x) dx - \frac{2\pi}{a^2} f(a)$</td>
</tr>
<tr>
<td>Shifted impulse</td>
<td>$\delta(r - a)$</td>
<td>$2\pi \nu J_0(2\pi \nu a)$</td>
</tr>
<tr>
<td>Exponential delay</td>
<td>$e^{-\nu r}$</td>
<td>$\frac{2\pi \nu}{(2\pi a^2 + \nu^2)^{3/2}}$</td>
</tr>
<tr>
<td>$e^{-r}$</td>
<td>$\frac{1}{\sqrt{2\pi a^2 q}}$</td>
<td>$\frac{(2\pi a^2 + \nu^2)^{1/2}}{2\pi \nu}$</td>
</tr>
<tr>
<td>$e^{-\nu^2 r^2}$</td>
<td>$\frac{1}{\sqrt{\pi a^2}}$</td>
<td>$\frac{1}{\sqrt{\pi a^2}} e^{-\nu^2}$</td>
</tr>
<tr>
<td>$\sin(2\pi r)$</td>
<td>$\frac{1}{2\nu} f(q/2\nu) , dq$</td>
<td></td>
</tr>
</tbody>
</table>

### Table 10-5 PROPERTIES OF THE HANKEL TRANSFORM

<table>
<thead>
<tr>
<th>Property</th>
<th>Spatial domain</th>
<th>Frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition theorem</td>
<td>$f(r) + g(r)$</td>
<td>$F(q) + G(q)$</td>
</tr>
<tr>
<td>Similarity theorem</td>
<td>$f(\alpha r)$</td>
<td>$\frac{1}{\alpha} F(\frac{q}{\alpha})$</td>
</tr>
<tr>
<td>Convolution theorem</td>
<td>$\int_0^\infty f(\nu) e^{i\nu r} d\nu \cdot e^{-i\nu q} d\nu = F(q)G(\nu)$</td>
<td></td>
</tr>
<tr>
<td>Convolution theorem</td>
<td>$\int_0^\infty f(\nu) g(\nu) , d\nu$</td>
<td>$F(q)G(q)$</td>
</tr>
<tr>
<td>Laplacian</td>
<td>$\nabla^2 f(r) = \frac{d^2 f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr}$</td>
<td>$-4\pi^2 q^2 F(q)$</td>
</tr>
<tr>
<td>Rayleigh's theorem</td>
<td>$\int_0^\infty [f(r)]^2 r , dr = E$</td>
<td>$\int_0^\infty [F(q)]^2 q , dq = E$</td>
</tr>
<tr>
<td>Power theorem</td>
<td>$\int_0^\infty [f(r)]^2 r , dr = P$</td>
<td>$\int_0^\infty [F(q)]^2 q , dq = P$</td>
</tr>
</tbody>
</table>

### 10.4.5.2 Computing the Hankel Transform

The projection theorem gives us a simple way to compute the Hankel transform of a function, which is useful, for example, in the study of optical systems, which commonly have circularly symmetric impulse responses and transfer functions. Eqs. (124), (125), (126) and (134) allow us to write...
Sec. 10.5 Correlation and the Power Spectrum

\[ F_r(q) = F(u, 0) = P(u) = \mathcal{F}\{p(x)\} \]  
(138)

and Eqs. (124), (127) and (128) imply that

\[ p(x) = \int_{-\infty}^{\infty} f_r(\sqrt{x^2 + y^2}) dy \]  
(139)

So

\[ F_r(q) = \mathcal{F}\left\{ \int_{-\infty}^{\infty} f_r(\sqrt{x^2 + y^2}) dy \right\} \]  
(140)

gives a two-step process for computing the Hankel transform: First project the function, and then compute its (one-dimensional) Fourier transform.

10.4.6 Interpretation

We conclude this introduction to the two-dimensional Fourier transform with Figure 10–10, which gives a bit of insight into the roles of amplitude and phase [11]. Parts (b) and (c) of the figure are displays of the amplitude and phase components, respectively, of the spectrum of the image in part (a).

One might be tempted to place more importance upon the amplitude spectrum, since it at least exhibits some recognizable structure, than upon the phase, which strikes the eye as essentially random. Eliminating the phase information, however, by setting the phase equal to zero and performing the inverse transformation yields part (d) of the figure—something bearing little resemblance to the original. On the other hand, eliminating the amplitude information (by setting the amplitude equal to a constant prior to the inverse transformation) yields part (e), a recognizable portrait.

While the amplitude spectrum specifies how much of each sinusoidal component is present, the phase information specifies where each of the sinusoidal components resides within the image. Figure 10–10 illustrates that disrupting this placement can create a devastating effect. As long as the components are kept in position, however, their amplitude appears to be less critical to the integrity of the image. For these reasons, most practical filters affect amplitude only, doing little or nothing to the phase information in the spectrum.

10.5 CORRELATION AND THE POWER SPECTRUM

In this section, we develop a series of analytical tools useful for studying the effects of noise in a linear system.

10.5.1 Autocorrelation

Recall that the self-convolution of a function is

\[ f(t) \ast f(t) = \int_{-\infty}^{\infty} f(t) f(t - t) dt \]  
(141)

If we do not reflect one term in the product, we form instead the autocorrelation function.
Sec. 10.7 Summary of Important Points

\[ R_f(t) = f(t) * f(-t) = \int_{-\infty}^{\infty} f(t) f(t + \tau) d\tau \]  
(142)

The autocorrelation function is always even and has a maximum at \( t = 0 \). It has the property

\[ \int_{-\infty}^{\infty} R_f(\tau) d\tau = \left[ \int_{-\infty}^{\infty} f(t) dt \right]^2 \]  
(143)

Every function has a unique autocorrelation function, but the converse is not true.

10.5.2 The Power Spectrum

The Fourier transform of the autocorrelation function is

\[ P_x(\omega) = \mathcal{F} \{ R_f(t) \} = \mathcal{F} \{ f(t) * f(-t) \} = f(\omega)F(\omega) \]  
(144)

and is called the power spectral density function or power spectrum of \( f(t) \). If \( f(t) \) is real, its autocorrelation function is real and even, and therefore, its power spectrum is real and even. Again, any \( f(t) \) has a unique power spectrum, but the converse is not the case.

10.5.3 Cross-correlation

Given two functions \( f(t) \) and \( g(t) \), their cross-correlation function is given by

\[ R_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)g(t + \tau) dt \]  
(145)

in a sense, the cross-correlation function indicates the relative degree to which two functions agree for various amounts of misalignment (shifting).

The Fourier transform of the cross-correlation function is the cross power spectral density function or cross power spectrum

\[ P_{fg}(\omega) = \mathcal{F} \{ R_{fg}(t) \} \]  
(146)

10.6 SUMMARY OF FOURIER TRANSFORM PROPERTIES

In this chapter, we have developed a number of properties of the Fourier transform that will prove useful in subsequent analyses of image-processing systems. For convenience of reference, these properties are summarized in Table 10-6.

10.7 SUMMARY OF IMPORTANT POINTS

1. The Fourier transform is a linear integral transformation that establishes a unique correspondence between a complex-valued function (e.g., of time) and a complex-valued function of frequency.
2. The Fourier transform of a Gaussian function is another Gaussian.
3. Evenness and oddness are preserved by the Fourier transform.
4. The Fourier transform of a real function is a Hermitian function.
5. The Fourier transform of a sum of functions is the sum of their individual transforms (addition theorem).
6. Shifting the origin of a function introduces into its spectrum a phase shift that is linear with frequency and that alters the distribution of energy between the real and imaginary parts of the spectrum without changing the total energy (shift theorem).
7. Convolution of two functions corresponds to multiplication of their Fourier transforms (convolution theorem).
8. Narrowing a function broadens its Fourier transform and vice versa (similarity theorem).
9. The energy of a function (signal) is the same as that of its Fourier transform (spectrum).
10. The transfer function of a linear system can be determined as the ratio of its (measured) output spectrum to its (known) input spectrum.
11. The Fourier transform of a sinusoidal function is an equally spaced impulse pair.
12. An input signal can be decomposed into an infinite sum of infinitesimal sinusoids.
13. A linear system can be thought of as operating separately on the sinusoidal components of the input signal, which are summed at the output to form the output signal.
14. The Fourier transform generalizes readily to functions of two or more dimensions.

15. If a function of two variables can be separated into a product of two functions of a single variable, then so can its Fourier transform.

16. Rotating a function of two dimensions rotates its Fourier transform by the same amount.

17. Projecting (collapsing) a two-dimensional function onto a line at an angle $\theta$ to the $x$-axis and transforming the resulting one-dimensional function yields a profile of the two-dimensional spectrum taken along a line at an angle $\theta$ to the $x$-axis.

18. Circularly symmetric two-dimensional functions have circularly symmetric spectra.

19. The Hankel transform relates the profile function of a circularly symmetric function to that of its spectrum.

20. Auto-correlation is self-convolution without reflection of either function.

21. Cross-correlation is like convolution, except that neither function is reflected.

22. The Fourier transform of the auto-correlation function is the power spectrum.

PROBLEMS

1. Illustrate graphically that the convolution of an even and an odd function produces an odd function.

2. Use integration by parts to prove the differentiation property in Table 10-6.

3. Suppose you have a TV camera that you suspect has a barely perceptible interface problem. You are convinced you can see, on hard-copy prints of digitized images, that every other line is slightly darker than the intervening lines. The manufacturer's representative says there's nothing wrong with the camera. How can you prove that there is a problem? You have a system capable of digitizing a TV image, averaging lines or columns of pixels and displaying a one-dimensional FFT. Describe the experiment and sketch the expected results.

4. Suppose you have two TV cameras that look identical, differing only in serial number. One is a special high-resolution model intended for a military customer, and the other is an economy model destined for a remote baby-sitting application. Due to a mix-up in shipping records, you don't know which is which. How can you identify the military camera? You have a system capable of digitizing a TV image, averaging lines or columns of pixels and displaying a one-dimensional FFT. Describe the experiment and sketch the expected results.

5. You have an RS-170 TV camera (see Figure 2-10) that has just been returned for repair. The customer says it has a problem with 60-Hz noise from the power line getting into the video signal. How can you verify that this is truly the problem before sending the camera out for repair? You have a system capable of digitizing a TV image, averaging lines or columns of pixels and displaying a one-dimensional FFT. Describe the experiment and sketch the expected results. Will the interlaced scan complicate the situation in this case, or not? Explain why or why not.

6. Suppose you have an RS-170 TV camera (see Figure 2-10) that has just been repaired. It had a problem with 40-KHz noise from the internal power supply getting into the video signal. How can you verify that the problem has indeed been fixed before placing the camera back into use? You have a system capable of digitizing a TV image, averaging lines or columns of pixels and displaying a one-dimensional FFT. Describe the experiment and sketch the expected results. Will the interlaced scan complicate the situation in this case? Explain why or why not.

PROJECTS

1. Develop a program that takes a single horizontal scan line out of a digital image and computes and displays a plot of the line's one-dimensional Fourier transform (amplitude and phase spectra). Use the program on a digital image of a tapered vertical bar to demonstrate the similarity theorem.

2. Develop a program as in Project 1, and add the capability to modify the amplitude spectrum (e.g., set a band of frequencies to zero), compute the inverse transform, plot the line, and rescan it onto the displayed image. Use the program to remove the high-frequency noise from a portion of a digital image.

3. Develop a program that can compute and display the two-dimensional Fourier transform (amplitude and phase spectra) of a digital image. Use the program on three digital images of the same scene taken through a wire screen held in front of the camera. Ensure that the screen is in focus well enough to be visible in the image. Rotate the screen $30^\circ$ between scans. Identify the components of the amplitude spectrum that are due to the screen.

4. Develop a program as in Project 3, and add the capability to modify the amplitude spectrum (e.g., set an annular region of frequencies to zero), compute the inverse transform, and display the image. Use the program to remove the high-frequency noise from a portion of a digital image.

5. Use a program as in Project 4 to remove the shading from a digitized image.

REFERENCES

For references to additional material on the Fourier transform, see Appendix 2.


