Epipolar Two-View Geometry
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Location of Mike seen by Bob
Location of Bob seen by Mike
Bob’s view
Mike’s view
Location of Mike seen by Bob

Bob's view

Location of Bob seen by Mike

Mike's view

Epipole
Given a point in Bob’s view, there exists a conjugate line passing the corresponding point in Mike’s view.

Observation:
Point correspondence

Bob’s view

Mike’s view
Point correspondence

Bob’s view

Mike’s view

8 correspondences
$P_1 = K[I_{3 \times 3} \ 0]$

Bob

$P_2 = K[R \ t]$

Mike
$X_1: 3D \text{ point in camera}$

$P_1 = K[I_{3 \times 3} \ 0]$

Bob

$P_2 = K[R \ t]$

Mike
\[ P_1 = K \begin{bmatrix} I_{3 \times 3} & 0 \end{bmatrix} \]

Bob

\[ P_2 = K \begin{bmatrix} R & t \end{bmatrix} \]

Mike

\( X_1 \): 3D point in camera 1

\( t \): Camera translation in camera 2
$X_2 = RX_1 + t$

3D point in camera 2

$t$ : Camera translation in camera 2

$P_1 = K[I_{3\times3} \ 0]$  
Bob

$P_2 = K[R \ t]$  
Mike
$X_2 = RX_1 + t$

3D point in camera 2

$t$: Camera translation in camera 2

$P_1 = K[I_{3 \times 3} \ 0]$

Bob

$P_2 = K[R \ t]$

Mike
$X_2 = RX_1 + t$

$X_2 - t = RX_1$

$t$: Camera translation in camera 2

$P_1 = K[I_{3x3} \ 0]$

Bob

$P_2 = K[R \ t]$

Mike
\[ X_2 = RX_1 + t \]

3D point in camera 2

\[ X_2 - t = RX_1 \]

Plane spanned by \( t \) and \( X_2 \).

\[ t \times X_2 = [t]_\times X_2 \]

\( t \) : Camera translation in camera 2

\[ P_1 = K[I_{3\times3} 0] \]

Bob

\[ P_2 = K[R \ t] \]

Mike
\[ 0 = (X_2 - t)^T [t]_x X_2 \]

\[ X_2 - t = RX_1 \]

3D point in camera 2

Plane spanned by \( X_2 \) and \( t \).

\[ t \times X_2 = [t]_x X_2 \]

\( t \): Camera translation in camera 2

\[ P_1 = K[I_{3\times3} \\ 0] \]

Bob

\[ P_2 = K[R \\ t] \]

Mike
\[ 0 = (X_2 - t)^T [t]_x X_2 = (RX_1)^T [t]_x X_2 \]

\[ X_2 - t = RX_1 \]

\[ X_2 = RX_1 + t \]

3D point in camera 2

\[ t \times X_2 = [t]_x X_2 \]

\[ t : \text{Camera translation in camera 2} \]

\[ P_1 = K[I_{3\times3} 0] \]

Bob

\[ P_2 = K[R t] \]

Mike
\[
0 = (X_2 - t)^T [t] X_2 = (RX_1)^T [t] X_2 \\
= X_1^T R^T [t] X_2
\]

\[
X_2 = RX_1 + t
\]

3D point in camera 2

Plane spanned by \( X_2 \) and \( t \times X_2 = [t] X_2 \)

\( t \): Camera translation in camera 2

\[
P_1 = K[I_{3 \times 3} \ 0]
\]
Bob

\[
P_2 = K[R \ t]
\]
Mike
0 = (X_2 - t)^T[t]_x X_2 = (RX_1)^T[t]_x X_2
= X_1^T R^T[t]_x X_2
= -X_2^T[t]_x RX_1 = -X_2^T EX_1

E = [t]_x R

X_2 - t = RX_1

X_2 = RX_1 + t

3D point in camera 2

Plane spanned by X_2 and X_1

\( t \times X_2 = [t]_x X_2 \)

\( t \) : Camera translation in camera 2

P_1 = K[ I_{3x3} \ 0 ]

Bob

P_2 = K[ R \ t ]

Mike
Observation:
Given a point in Bob’s view, there exists a conjugate line passing the corresponding po
\[ X_2^T E X_1 = 0 \]

\[ E = [t]_x R \]

\[ P_1 = K[I_{3 \times 3} \ 0] \]

\[ P_2 = K[R \ t] \]

Bob Mike
\[ X_2^T E X_1 = 0 \]
\[ E = [t]_x R \]

\[ P_1 = K[I_{3 \times 3} 0] \]
\[ P_2 = K[R t] \]
$\mathbf{P}_1 = K[I_{3 \times 3} \ 0]$

$\mathbf{P}_2 = K[R \ t]$

$\mathbf{X}_2^T \mathbf{E} \mathbf{X}_1 = 0$

$\mathbf{E} = [t]_\times \mathbf{R}$
$X_2^T E X_1 = 0$

$E = [t] x R$

$P_1 = K[I_{3 \times 3} \ 0]$

$P_2 = K[R \ t]$
\[
\begin{align*}
\mathbf{X}_2^T \mathbf{E} \mathbf{X}_1 &= 0 \\
\mathbf{E} &= \begin{bmatrix} \mathbf{t} \end{bmatrix}_x \mathbf{R}
\end{align*}
\]

\[
\mathbf{P}_1 = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & 0 \end{bmatrix} \\
\text{Bob}
\]

\[
\mathbf{P}_2 = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \\
\text{Mike}
\]

\[\text{Epipolar line}\]
Epipolar line computation

\[ x_1 = \lambda_1 X_1 \]

\[ X_2^T E X_1 = 0 \]

\[ E = [t]_x R \]

\[ P_1 = K[I_{3\times3} \ 0] \]

Bob

\[ P_2 = K[R \ t] \]

Mike
Epipolar line computation

\[ x_1 = \lambda_1 X_1 \]
\[ X_2^T E X_1 = 0 \]
\[ E = [t]_x R \]
\[ x_2 = \lambda_2 X_2 \]

\[ P_1 = K[I_{3	imes3} \ 0] \]
Bob

\[ P_2 = K[R \ t] \]
Mike
Epipolar line computation

$x_1 = \lambda_1 X_1$

$x_2^T E x_1 / \lambda_1 \lambda_2 = 0$

$E = [t]_x R$

$x_2 = \lambda_2 X_2$

$P_1 = K [I_{3x3} \ 0]$

Bob

$P_2 = K [R \ t]$

Mike
Epipolar line computation

\[ P_1 = K \begin{bmatrix} I_{3 \times 3} & 0 \end{bmatrix} \]
Bob

\[ P_2 = K \begin{bmatrix} R & t \end{bmatrix} \]
Mike

\[ x_2^T E x_1 = 0 \]

\[ E = [t]_x R \]
Epipolar line computation

\[ x_2^T Ex_1 = 0 \]

\[ P_1 = K \begin{bmatrix} I_{3 \times 3} & 0 \end{bmatrix} \]

Bob

\[ P_2 = K \begin{bmatrix} R & t \end{bmatrix} \]

Mike
Epipolar line computation

\[ P_1 = K \begin{bmatrix} I_{3 \times 3} & 0 \end{bmatrix} \]

Bob

\[ P_2 = K \begin{bmatrix} R & t \end{bmatrix} \]

Mike

\[ x_2^T E x_1 = 0 \]
Epipole computation

\[ x_2^T E x_1 = 0 \]
\[ E e_1 = 0 \]

Epipole

\[ P_1 = K [ I_{3 \times 3} \ 0 ] \]
Bob

\[ P_2 = K [ R \ t ] \]
Mike
Epipole computation

\[ x_2^T E x_1 = 0 \]
\[ e_2^T E = 0 \]

Epipole

\[ P_1 = K \begin{bmatrix} I_{3 	imes 3} & 0 \end{bmatrix} \]
Bob

\[ P_2 = K \begin{bmatrix} R & t \end{bmatrix} \]
Mike
How can we compute the E-matrix?

If

$$E = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}$$

then epipolar constraint can be rewritten as

$$q^T \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = q^T (p_x e_1 + p_y e_2 + p_z e_3)$$

$$= (p_x q^T \quad p_y q^T \quad p_z q^T) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0$$

This equation is linear and homogeneous in $E' = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$. 
The 8-point algorithm

Let \( \vec{a} = (pxq^T \quad pyq^T \quad pzq^T) \)

\[
\begin{pmatrix}
  a_1^T \\
  a_2^T \\
  \vdots \\
  a_n^T
\end{pmatrix} \quad E' = 0
\]

where \( a_i \) is the known \( 1 \times 9 \) vector of image points and \( E' \) is the essential matrix re-organized into a \( 9 \times 1 \) column vector.

\( E' \) has to be in the null-space of \( \begin{pmatrix}
  a_1^T \\
  a_2^T \\
  \vdots \\
  a_n^T
\end{pmatrix} \).
Properties of the Essential matrix

\[
E^T = R^T \hat{T} T = 0
\]

E is a singular matrix, \( \det(E) = 0 \).

\[
EE^T = \hat{T} \hat{T}^T
= \begin{bmatrix}
  t_x^2 & t_xt_y & t_xt_z \\
  t_xt_y & t_y^2 & t_yt_z \\
  t_xt_z & t_yt_z & t_z^2
\end{bmatrix}
- \|T\|^2 I
\]
Properties of the Essential matrix

\[ EE^T = \hat{T} \hat{T}^T = \hat{T} \hat{T}^T - \hat{T}^T \hat{T} \hat{T}^T \]

\[
= \begin{bmatrix}
    t_x^2 & t_xt_y & t_xt_z \\
    t_xt_y & t_y^2 & t_yt_z \\
    t_xt_z & t_yt_z & t_z^2 \\
\end{bmatrix} - \|T\|^2 I
\]

If we solve the characteristic polynomial \( \det(EE^T - \lambda I) = 0 \) we will find two eigenvalues both equal to \( \|T\|^2 \).
Properties of the Essential matrix

Recall that the singular values of $E$ are the square-roots of the eigenvalues of $EE^T$ if $E$ is a square matrix.

Hence, we have proved that if a matrix is essential, namely, can be decomposed as the product of an antisymmetric $\hat{T}$ and a special orthogonal $R$ then its singular values are $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$.

This is helpful in checking if a matrix is essential but is not constructive on how to decompose it.
Properties of the Essential matrix

We have to prove the sufficient condition:

If the singular values of a matrix are are $\sigma_1 = \sigma_2 > 0$ and $\sigma_3 = 0$ then the matrix can be decomposed into the product of an antisymmetric $\hat{T}$ and a special orthogonal $R$. 
We need a lemma!

If $Q$ is orthogonal ($Q^T Q = I$), then

$$\hat{Q}a = Q\hat{a}Q^T$$

**Proof:** $\hat{Q}ab = Qa \times b = Q(a \times Q^T b) = Q\hat{a}Q^T b$.

and the following simple fact

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_z^T R_{z,\pi/2}$$
We need a lemma!

If $Q$ is orthogonal ($Q^TQ = I$), then

$$\hat{Q}a = Q\hat{a}Q^T$$

**Proof:** $\hat{Q}ab = Qa \times b = Q(a \times Q^Tb) = Q\hat{a}Q^Tb$.

and the following simple fact

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T_z^T R_{z,\pi/2}$
\[ E = U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T \]

\[ = \sigma U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T \]

\[ = \sigma \underbrace{U T_z}^T \underbrace{R_z V^T}_{\text{antisymmetric orthogonal}} \]

Observe \( U T_z = U \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \), which is the last column of \( U \)
**Necessary and sufficient condition:** $E$ is essential iff

$$\sigma_1(E) = \sigma_2(E) \neq 0 \text{ and } \sigma_3(E) = 0.$$
We just showed that there is at least one such decomposition $\hat{T}R$, but is it unique?

We showed the following decomposition:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
= \hat{T}_z R_{z, \pi/2}
\]

But we could similarly write

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
= \hat{T}_z R_{z, -\pi/2}.
\]
We just showed that there is at least one such decomposition $\widehat{T}R$, but is it unique?

We showed the following decomposition:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[= \widehat{T}_z \times R_{z,\pi/2} \]

But we could similarly write

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} = \widehat{T}_z R_{z,-\pi/2}.
\]
If \( E = U \Sigma V^T = U \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T \), there are four solutions for the pair \((\hat{T}, R)\):

\[
\begin{align*}
(\hat{T}_1, R_1) &= (U R_{z,+\pi/2} \Sigma U^T, U R^T_{z,+\pi/2} V^T) \\
(\hat{T}_2, R_2) &= (U R_{z,-\pi/2} \Sigma U^T, U R^T_{z,-\pi/2} V^T) \\
(\hat{T}_1, R_2) &= (U R_{z,+\pi/2} \Sigma U^T, U R^T_{z,-\pi/2} V^T) \\
(\hat{T}_2, R_1) &= (U R_{z,-\pi/2} \Sigma U^T, U R^T_{z,+\pi/2} V^T)
\end{align*}
\]

Please remember that we have to make \( R \) have \( \det(R) = 0 \), see our Procrustes lecture.
**Mirror ambiguity:** If $T$ is a solution, then $-T$ is a solution, too. There is no way to disambiguate from the epipolar constraint: $q^T(-T \times R_p) = 0$.

**Twisted pair ambiguity:** If $R$ is a solution, then also $R_{T,\pi}R$ is a solution. The first image is “twisted” around the baseline 180 degrees.
**Mirror ambiguity:** If $T$ is a solution, then $-T$ is a solution, too. There is no way to disambiguate from the epipolar constraint: $q^T(-T \times R_p) = 0$.

**Twisted pair ambiguity:** If $R$ is a solution, then also $R_{T,\pi} R$ is a solution. The first image is “twisted” around the baseline 180 degrees.
The full two-view algorithm

1. Build the homogeneous linear system by stacking epipolar constraints
   \[ q_i^T (T \times Rp_i) = 0, \quad i = 1, \ldots, 8: \]
   \[
   \begin{bmatrix}
   \vdots \\
   (q_i \otimes p_i)^T \\
   \vdots \\
   \\
   \end{bmatrix}
   \begin{bmatrix}
   e'_1 \\
   e'_2 \\
   e'_3 \\
   \end{bmatrix}
   = A \ (8 \times 9)
   \]

2. Let
   \[
   \begin{bmatrix}
   e'_1 \\
   e'_2 \\
   e'_3 \\
   \end{bmatrix}
   \]
   be the nullspace of \( A \) (if \( \sigma_8 \approx 0 \) give up)
The full two-view algorithm

1. Build the homogeneous linear system by stacking epipolar constraints
   \[ q_i^T (T \times R p_i) = 0, \quad i = 1, \ldots, 8: \]
   \[
   \begin{bmatrix}
   (q_i \otimes p_i)^T \\
   \vdots \\
   \end{bmatrix}
   \begin{bmatrix}
   e_1' \\
   e_2' \\
   e_3'
   \end{bmatrix}
   \]
   \[ A (8 \times 3) \]

2. Let \[ \begin{bmatrix}
   e_1' \\
   e_2' \\
   e_3'
   \end{bmatrix} \] be the nullspace of \( A \) (if \( \sigma_8 \approx 0 \) give up)
The full two-view algorithm

$\begin{bmatrix} e'_1 & e'_2 & e'_3 \end{bmatrix} = U \text{diag} (\sigma'_1, s'_2, \sigma'_3) V^T$. Then use the following estimate of the essential matrix:

$$E = U \text{diag} \left( \frac{\sigma'_1 + \sigma'_2}{2}, \frac{\sigma'_1 + \sigma'_2}{2}, 0 \right) V^T$$

$T = \pm \hat{u}_3 \quad R = U R_{Z,\pi/2} V^T$ or $R = R_{T,\pi} R$

Try all four pairs $(T, R)$ to check if reconstructed points are in front of the cameras $\lambda q = \mu R p + T$ give $\lambda, \mu > 0$. 
The full two-view algorithm

3 \[ \begin{bmatrix} e'_1 & e'_2 & e'_3 \end{bmatrix} = U \text{diag } (\sigma'_1, s'_2, \sigma'_3) V^T. \] Then use the following estimate of the essential matrix:

\[ E = U \text{diag } \left( \frac{\sigma'_1 + \sigma'_2}{2}, \frac{\sigma'_1 + \sigma'_2}{2}, 0 \right) V^T \]

4 \[ T = \pm \hat{u}_3 \quad R = UR_{Z, \pi/2}V^T \quad \text{or} \quad R = R_{T, \pi}R \]

5 Try all four pairs \((T, R)\) to check if reconstructed points are in front of the cameras \[ \lambda q = \mu Rp + T \] give \(\lambda, \mu > 0\).
Triangulation is possible if we have computed $R$ and $T$ but again up to a scale factor. Set $\|T\| = 1$:

$$(q_i - Rp_i) \begin{pmatrix} \mu_i \\ \lambda_i \end{pmatrix} = T$$

There are then 3 equations with 2 unknowns $\lambda_i$ and $\mu_i$ for each point.

Solve with pseudo-inverse.