What can we learn from vanishing points

- Two vanishing points give us information about camera orientation with respect to a plane.
- Three vanishing points from three orthogonal parallel sets ("cube") reveals information about camera to world coordinates.
- One vanishing points reveals partial information about where is projection center.
- Three “orthogonal” vanishing points reveal information about image center (u0,v0) and focal length.
A and B reveal the first two columns of rotation to world

$K^{-1}A$ is the x-axis of the world in camera coordinates

$K^{-1}B$ is the y-axis of the world in camera coordinates
If we connect two vanishing points we obtain the horizon with line equation \((K^{-1}A \times K^{-1}B)^T \rho = 0\).
Horizon with projection center build a horizon plane with the same normal as the world plane!

Horizon

Projection center

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The horizon plane is parallel to the ground plane and hence $K^{-1}A \times K^{-1}B$ is the normal to the ground plane expressed via pixels.

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Horizon gives complete info about how ground plane is oriented! If horizon is horizontal the center it means that optical axis is parallel to groundplane!
Horizon gives complete info about how ground plane is oriented! If horizon is horizontal the center it means that optical axis is parallel to ground-plane!
If horizon moves to the bottom it means we look upwards!
If horizon moves to the bottom it means we look upwards!
If horizon moves to the top it means we look downwards!
If horizon moves to the bottom it means we look upwards!
Three orthogonal vanishing points
Z direction in the world coordinate system
Z direction in the world coordinate system
\[ z_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T \]

z point at infinity

Z direction in the world coordinate system

\( v_z : z \text{ vanishing point} \)
Columns of the rotation matrix represent vanishing points of world axes.

\[
zv_z = K \begin{bmatrix} r_1 & r_2 & r_3 & t \end{bmatrix} z_\infty
\]

\( z \) vanishing point
\( z \) point at infinity

\[
R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in SO(3)
\]
Columns of the rotation matrix represent vanishing points of world axes.

\[
zv_z = K \begin{bmatrix} r_1 & r_2 & r_3 & t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
\end{bmatrix}
\]

\[
R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in SO(3)
\]

\[
z_\infty = [0 \ 0 \ 1 \ 0]^T
\]

Z direction in the world coordinate system

v = K \cdot r 

v_z : z vanishing point

v_z : z point at infinity
\[ \mathbf{z}_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T \]

- \( \mathbf{z}_\infty \) point at infinity

- Columns of the rotation matrix represent vanishing points of world axes.

- \( r_3 = \mathbf{K}^{-1} \mathbf{v}_z / \| \mathbf{K}^{-1} \mathbf{v}_z \| \)

- \( \mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{SO}(3) \)
Geometric interpretation

\[ \mathbf{v}_z = \mathbf{T}_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1} & 0 \end{bmatrix} \]

\[ \mathbf{z}_\infty = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T \]

Z direction in the world coordinate system

\[ \mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathbb{SO}(3) \]

Camera axes

World z-axis

\[ \beta \]

\[ \alpha \]

z-point at infinity

\[ \mathbf{v}_z : z \text{ vanishing point} \]

u, v

z-axis of world coordinate system
Geometric interpretation

\[ Z_{\infty} = [0 \ 0 \ 1 \ 0]^T \]

\( z \) point at infinity

\( z \) direction in the world coordinate system

**Camera axes**

**World z-axis**

\[ r_3 = \frac{K^{-1}v_z}{\| K^{-1}v_z \|} \]

\[ = \begin{bmatrix} \sin \alpha \sin \beta \\ \cos \beta \\ \cos \alpha \sin \beta \end{bmatrix} \]

**World z-axis**

\( z \) axis of world coordinate system

\( v_z : z \) vanishing point

\( R = [r_1 \ r_2 \ r_3] \in SO(3) \)

**Geometric interpretation**

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Geometric interpretation

\[ z_{\infty} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T \]

Z direction in the world coordinate system

\[ \mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \text{SO}(3) \]

Pan and tilt angles

\[ \alpha = \tan^{-1}(r_3(1)/r_3(3)) \]
\[ \beta = \cos^{-1}r_3(2) \]

Camera axes

World z-axis

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Case 2: Using two vanishing points

First person coordinate system

World coordinate system

Towards vanishing point

$R \in SO(3), t \in \mathbb{R}^3$

How to recover?

$x_{\infty} = [1 \ 0 \ 0 \ 0]^T$ x point at infinity

Columns of the rotation matrix represent vanishing points of world axes.

$r_3 = K^{-1} z v_z$

$r_1 = K^{-1} z v_x$

$r_2 = r_3 \times r_1$

Orthogonal rotation matrix $R = [r_1 \ r_2 \ r_3] \in SO(3)$

$z_{\infty} = [0 \ 0 \ 1 \ 0]^T$ z point at infinity

$R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in SO(3)$
Calibration via Vanishing Points

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Calibration via Vanishing Points

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\[ r_3 = K^{-1} z v_z \]
\[ r_1 = K^{-1} z v_x \]
\[ r_2 = r_3 \times r_1 \]
\[ R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \]
Exercise I

\[ r_1 = K^{-1}zv_x \]
\[ r_2 = K^{-1}zv_y \]

\[ R = \begin{bmatrix} r_1 & r_2 & r_1 \times r_2 \end{bmatrix} \]
Exercise II

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Exercise II

\[ r_1 = K^{-1}v_x / \|K^{-1}v_x\| \]
\[ r_2 = K^{-1}v_y / \|K^{-1}v_y\| \]

Scale normalization

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Exercise II

\[ r_1 = (0.8017, -0.2086, 0.5602)^T \]
\[ r_2 = (0.0067, 0.9411, 0.3382)^T \]
\[ r_3 = r_1 \times r_2 = (-0.5988, -0.2673, 0.7558)^T \]
Exercise II

Estimate pan/tilt from $r_3$.

$$\alpha = \tan^{-1}(r_3(1)/r_3(3))$$
$$\beta = \sin^{-1}r_3(2)$$

$$\alpha = -0.6691 = -0.2130\pi$$
$$\beta = -0.2706 = -0.0861\pi$$

$$R = \begin{pmatrix}
0.8017 & 0.0067 & -0.5977 \\
-0.2086 & 0.9411 & -0.2673 \\
0.5602 & 0.3382 & 0.7558 \\
\end{pmatrix}$$

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Exercise II
Planar world

\[ m = [u \ v \ 1]^T \]

\[ X = [X \ Y \ 0 \ 1] \]

\[ P = [R \ t] \]

\[ zm = K[r_1 \ r_2 \ r_3 \ | \ t]X \]

\[ = K[r_1 \ r_2 \ r_3 \ | \ t] \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} = K[r_1 \ r_2 \ t] \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \]

2D homography

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Planar world

\[ X = [X \ Y \ 0 \ 1] \]

\[ m = [u \ v \ 1]^T \]

\[ P = [R \ t] \]

where \( H = K \begin{bmatrix} r_1 & r_2 & t \end{bmatrix} \)
Exercise

Homography from four points:

\[ H = \begin{pmatrix} 0.4430 & 0.0037 & -0.1071 \\ -0.1153 & 0.5216 & 0.1506 \\ 0.3096 & 0.1875 & 0.5944 \end{pmatrix} \]

\[ H = K^{-1} H = \begin{bmatrix} r_1 & r_2 & t \end{bmatrix} \]

Note that \( \| r_1 \| = \| r_2 \| = 1 \)
Exercise

Homography from four points:

\[ H = \begin{pmatrix} 0.4430 & 0.0037 & -0.1071 \\ -0.1153 & 0.5216 & 0.1506 \\ 0.3096 & 0.1875 & 0.5944 \end{pmatrix} \]

\[ a = \| (H_{11}, H_{21}, H_{31}) \| \quad \text{: Normalization factor} \]

\[ t = H(:, 3)/a = (-0.1937, 0.2726, 1.0756)^T \]

\[ r_1 = H(:, 1)/a = (0.8017, -0.2086, 0.5602)^T \]

\[ r_2 = H(:, 2)/a = (0.0067, 0.9439, 0.3392)^T \]
Exercise

Homography from four points:

$$H = \begin{pmatrix}
0.4430 & 0.0037 & -0.1071 \\
-0.1153 & 0.5216 & 0.1506 \\
0.3096 & 0.1875 & 0.5944 
\end{pmatrix}$$

$$a = \left\| (H_{11}, H_{21}, H_{31}) \right\|$$: Normalization factor

$$t = H(:, 3)/a = (-0.1937, 0.2726, 1.0756)^T$$

$$r_1 = H(:, 1)/a = (0.8017, -0.2086, 0.5602)^T$$

$$r_2 = H(:, 2)/a = (0.0067, 0.9439, 0.3392)^T$$

$$r_3 = r_1 \times r_2 = (-0.1937, 0.2726, 1.0756)^T$$
How to constraint the projection center using point at infinity
• Intersecting lines in the world appear as parallel in the image if the intersection point is projected to infinity!
• The opposite than the vanishing point.
• At this position we know that the projection center lies on a plane through the world intersection point.
Perception:
How to compute intrinsics from vanishing points

Kostas Daniilidis
Manhattan world: A scene with three orthogonal sets of parallel lines

Three orthogonal sets of parallel lines create three orthogonal vanishing points
Line connecting AB is the horizon!

Remember that the horizon gives us the orientation of the ground plane with respect to the camera!
Obvious question: If the horizon AB gives us information about the ground plane and C corresponds to the vertical then shouldn’t be C determined by AB?

The answer is no because we omitted the influence of the focal length and the image center.
Let’s look at ABC as a tetrahedron OABC incl the projection center

O is Projection center
Let H be the orthocenter of the triangle ABC.

Theorem from Euclidean Geometry:
If H is the orthocenter of ABC and all three angles AOB, BOC, and COA are right angles, the OH is perpendicular to ABC plane.

OH is the optical axis and ABC is the image plane, hence, H is the image center.
We found the image center! What about the focal length \((f=OH)\)? Can it be computed from \(A, B, \text{ and } C\)?

Simple 9th grade geometry:
- \(h^2 = d_1d_2\) and
- \(f^2 + d_3^2 = h^2\) (Pythagoras)

Hence
- \(f^2 = d_1d_2 - d_3^2\)
Three orthogonal vanishing points allow computation of focal length and image center!