Sampling: Converting analog time or space to digital
Fidget spinners!
Back to the sine-wave: sampling interval 1/8 of wavelength

For note A (440 Hz), sampling frequency of 44.1kHz means 100 samples per wavelength!
Increasing sampling interval to 1/4 of wavelength
Increasing sampling interval to more than $\frac{1}{2}$ of wavelength

We cannot reconstruct the original wave!
What does sampling mean?

Sampling is a multiplication of the signal by the comb function $\Pi_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. $T$ is the sampling interval:

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

(S for “sampled”) After sampling, we forget about the $T$: we just obtain a sequence of numbers, that we note $f_s[k]$. 

\[ X \times \text{ } = \text{ } \]
Multiplication in time means convolution in frequency!

Question 1: what is the Fourier transform of the comb?

A comb with inverse sampling interval!
Question 2: What means convolution with a comb?

\[ F_S(\omega) = F(\omega) \ast \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi n}{T}) \]

It is creating copies of the "spectrum" at inverse time intervals!
Question 3: Can we reconstruct the original spectrum from the “xeroxed” spectrum?

Only if the copies do not overlap! Aliasing!
Shannon-Nyquist Theorem

An analog signal can be reconstructed from sampling only if the sampling frequency is at least twice the maximum frequency; equivalently, if the sampling interval is less equal the minimum half the minimum wavelength.

Obviously the analog signal has to be bandlimited, otherwise there is no maximum frequency!

\[ f_{\text{sampling}} \geq 2f_{\text{max}} \quad \text{or} \quad T_{\text{sampling}} \leq \frac{T_{\text{min}}}{2} \]
How can the reconstruction be realized?

Multiplying the spectrum of the sampled signal with the rectangle-function

$$\Pi(t) = \begin{cases} \frac{1}{2\pi} & -\pi \leq \omega \leq \pi \\ 0 & \text{anywhere else} \end{cases}$$
Which function has the box as Fourier-transform?

\[
\mathcal{F}^{-1}\{\text{rect}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{rect}(\omega) e^{j\omega t} dt
\]

\[
= \frac{1}{2\pi} \int_{-1/2}^{1/2} e^{j\omega t} dt
\]

\[
= \frac{1}{2\pi} \left[ \frac{1}{j\omega} e^{j\omega t} \right]_{-1/2}^{1/2}
\]

\[
= \frac{1}{2\pi} \text{sinc} \left( \frac{t}{2} \right)
\]

\[
\frac{1}{2\pi} \text{sinc}(t/2) \Rightarrow \text{rect}(\omega)
\]

\[
2\pi \text{sinc}(\pi t) \Rightarrow \frac{1}{2\pi} \text{rect} \left( \frac{\omega}{2\pi} \right)
\]
Multiplication in frequency means convolution in time (space)

Reconstruction means convolution of the sampled signal with sinc

\[
\begin{align*}
    f_{\text{reconstr}}(t) &= f(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) * 2\pi\text{sinc}(\pi t) \\
    &= \sum_{n=-\infty}^{\infty} f[n] 2\pi\text{sinc}(\pi t)
\end{align*}
\]
Discrete signals and Discrete Fourier Transform
Discrete time signals have still continuous Fourier transforms (CFT)!
We call it also DTFT (Discrete Time Fourier Transform).

\[
f[n] \rightarrow \sum_{n=0}^{L-1} f[n]e^{-j\omega n}
\]

Discrete time signals have continuous and periodic Fourier transforms with period \(2\pi\)!
That’s why we consider only the interval \((-\pi,\pi]\) when we talk about the CFT of a discrete signal!
Example. Let $f[n]$ a discrete time signal and $h[n]$ a discrete time signal defined as:

$$h[n] = \begin{cases} 
1, & n = 0 \\
-1, & n = 1 \\
0, & \text{elsewhere}
\end{cases}$$

$$H(\omega) = \sum_n h[n] e^{-j\omega n} = h[0] e^{-j\omega \cdot 0} + h[1] e^{-j\omega \cdot 1} = 1 - e^{-j\omega}$$
What does discrete frequency mean?
What does sampling in the frequency domain correspond to in the time/space domain? It corresponds to a **convolution**:

- **Frequency domain**: multiplication with $\sum \delta(\omega - \frac{2\pi k}{L})$

- **Time domain**: convolution with $\sum \delta[n - kL]$, equivalent to replication of the signal at multiples of its length.
How is the DFT of a discrete time finite signal different than the DTFT (CFT)?

Sampling of the frequency domain only means replicating the original signal.
Some MATLAB....

```matlab
support = 16;
sigma = 2;
d = -floor(support/2)+1:floor(support/2);
g = (1/(sigma*sqrt(2*pi)))*exp(-d.^2/(2*sigma^2));

figure(1)
stem(d,g);

figure;
stem(d,abs(fft(g)));
```
fftshift... performs a shift by L/2 in the frequency domain which is $e^{-j\pi L n / (2L)}$ in time domain corresponding to multiplying signal with $(-1)^n$ which means $-h[1], h[2], -h[3], ...$