Rotation
Rotation = axis + angle

Anders Adermark @ flickr

“30 minutes shutter time. How cool isn't this? You see the stars above the north axis of our globe being fixed, and then the circles gets wider and wider the further you get from them. “
Axis Angle Representation
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Euler’s theorem
Any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.
Axis Angle Representation

A quadruple can represent 3D rotation, i.e., \( q = (e, \theta) \), where \( \|e\| = 1 \)

Euler’s theorem
Any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.
“I wanted to have a water jet in my garden: Euler calculated the force of the wheels necessary to raise the water to a reservoir, from where it should fall back through channels, finally spurting out in Sanssouci. My mill was carried out geometrically and could not raise a mouthful of water closer than fifty paces to the reservoir. Vanity of vanities! Vanity of geometry!” - Frederick II
He can do geometry
Euler’s work

Graph Theory:

\[ V - E + F = 2 \]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \to \infty} \left( \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \right).
\]

\[
e^{i\pi} + 1 = 0
\]
Euler’s theorem: every spatial rotation has a rotation axis.

- Let $O, \mathbf{n}, \theta$, be . . .
- Let $\text{rot}(\mathbf{n}, \theta)$ be the corresponding rotation.
- Many to one:
  - $\text{rot}(-\mathbf{n}, -\theta) = \text{rot}(\mathbf{n}, \theta)$
  - $\text{rot}(\mathbf{n}, \theta + 2k\pi) = \text{rot}(\mathbf{n}, \theta)$, for any integer $k$.
  - So, restrict $\theta$ to $[0, \pi]$. But not smooth at the edges.
  - When $\theta = 0$, the rotation axis is indeterminate, giving an infinity-to-one mapping.
  - Again you can fix by adopting a convention for $\mathbf{n}$, but result is not smooth.
  - (Or, what about using the product, $\theta \mathbf{n}$? Later.)
What do we want from axis-angle?

- Operate on points
  - Rodrigues’s formula
- Compose rotations, average, interpolate, sampling, …?
  - Not using axis-angle
- Convert to other representations? There aren’t any yet. But, later we will use axis-angle *big time*. It’s very close to *quaternions*. 
Rodrigues’s formula

Others derive Rodrigues’s formula using rotation matrices: ugly and messy. The geometrical approach is clean and insightful.

- Given point $\mathbf{x}$, decompose into components parallel and perpendicular to the rotation axis

\[
\mathbf{x} = \mathbf{n}(\mathbf{n} \cdot \mathbf{x}) - \mathbf{n} \times (\mathbf{n} \times \mathbf{x})
\]

- Only $\mathbf{x}_\perp$ is affected by the rotation, yielding Rodrigues’s formula:

\[
\mathbf{x}' = \mathbf{n}(\mathbf{n} \cdot \mathbf{x}) + \sin \theta (\mathbf{n} \times \mathbf{x}) - \cos \theta \mathbf{n} \times (\mathbf{n} \times \mathbf{x})
\]

- A common variation:

\[
\mathbf{x}' = \mathbf{x} + (\sin \theta) \mathbf{n} \times \mathbf{x} + (1 - \cos \theta) \mathbf{n} \times (\mathbf{n} \times \mathbf{x})
\]
Rotation matrices

- Choose $O$ on rotation axis. Choose frame $(\mathbf{\hat{u}}_1, \mathbf{\hat{u}}_2, \mathbf{\hat{u}}_3)$.
- Let $(\mathbf{\hat{u}}'_1, \mathbf{\hat{u}}'_2, \mathbf{\hat{u}}'_3)$ be the image of that frame.
- Write the $\mathbf{\hat{u}}'_i$ vectors in $\mathbf{\hat{u}}_i$ coordinates, and collect them in a matrix:

\[
\begin{align*}
\mathbf{\hat{u}}'_1 &= \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \mathbf{\hat{u}}_1 \cdot \mathbf{\hat{u}}'_1 \\ \mathbf{\hat{u}}_2 \cdot \mathbf{\hat{u}}'_1 \\ \mathbf{\hat{u}}_3 \cdot \mathbf{\hat{u}}'_1 \end{pmatrix} \\
\mathbf{\hat{u}}'_2 &= \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} \mathbf{\hat{u}}_1 \cdot \mathbf{\hat{u}}'_2 \\ \mathbf{\hat{u}}_2 \cdot \mathbf{\hat{u}}'_2 \\ \mathbf{\hat{u}}_3 \cdot \mathbf{\hat{u}}'_2 \end{pmatrix} \\
\mathbf{\hat{u}}'_3 &= \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{\hat{u}}_1 \cdot \mathbf{\hat{u}}'_3 \\ \mathbf{\hat{u}}_2 \cdot \mathbf{\hat{u}}'_3 \\ \mathbf{\hat{u}}_3 \cdot \mathbf{\hat{u}}'_3 \end{pmatrix} \\
A &= (a_{ij}) = (\mathbf{\hat{u}}'_1 | \mathbf{\hat{u}}'_2 | \mathbf{\hat{u}}'_3)
\end{align*}
\]
So many numbers!

- A rotation matrix has nine numbers,
- but spatial rotations have only three degrees of freedom,
- leaving six excess numbers . . .
- There are six constraints that hold among the nine numbers.

\[
|\hat{u}_1'| = |\hat{u}_2'| = |\hat{u}_3'| = 1
\]
\[
\hat{u}_3' = \hat{u}_1' \times \hat{u}_2'
\]

- *i.e.* the $\hat{u}_i'$ are unit vectors forming a right-handed coordinate system.
- Such matrices are called *orthonormal* or *rotation* matrices.
Rotating a point

- Let \((x_1, x_2, x_3)\) be coordinates of \(x\) in frame \((\hat{u}_1, \hat{u}_2, \hat{u}_3)\).
- Then \(x'\) is given by the same coordinates taken in the \((\hat{u}'_1, \hat{u}'_2, \hat{u}'_3)\) frame:

\[
x' = x_1 \hat{u}'_1 + x_2 \hat{u}'_2 + x_3 \hat{u}'_3 \\
= x_1 A\hat{u}_1 + x_2 A\hat{u}_2 + x_3 A\hat{u}_3 \\
= A(x_1 \hat{u}_1 + x_2 \hat{u}_2 + x_3 \hat{u}_3) \\
= Ax
\]

- So rotating a point is implemented by ordinary matrix multiplication.
Sub- and superscript notation for rotating a point

- Let $A$ and $B$ be coordinate frames.
- Let $^A\!x$ be coordinates in frame $A$.
- Let $^B_A\!R$ be the rotation matrix that rotates frame $B$ to frame $A$.
- Then (see previous slide) $^B_A\!R$ represents the rotation of the point $x$:
  $$^B\!x' = ^B_A\!R \cdot ^B\!x$$
- Note presuperscripts all match. Both points, and xform, must be written in same coordinate frame.
Coordinate transform

There is another use for $^B_A R$:

- $^A x$ and $^B x$ represent the same point, in frames $A$ and $B$ resp.
- To transform from $A$ to $B$:
  
  $$^B x = ^B_A R ^A x$$

- For coord xform, matrix subscript and vector superscript “cancel”.

Rotation from $B$ to $A$ is the same as coordinate transform from $A$ to $B$. 

Matthew T. Mason
Nice things about rotation matrices

- Composition of rotations: \( \{R_1; R_2\} = R_2 R_1 \).
  \( \{x; y\} \) means do \( x \) then do \( y \).
- Inverse of rotation matrix is its transpose
  \( B_A R^{-1} = A_B R = A_B R^T \).
- Coordinate xform of a rotation matrix:
  \[ B_R = B_A R_A A_B R_B \]
Example rotation matrix

\[
\mathbf{B}_A^R = \begin{pmatrix} \mathbf{B}x_A & \mathbf{B}y_A & \mathbf{B}z_A \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
\]

How to remember what $\mathbf{B}_A^R$ does? Pick a coordinate axis and see. The $x$ axis isn’t very interesting, so try $y$:

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
Converting rot(\(\hat{n}, \theta\)) to \(R\)

- Ugly way: define frame with \(\hat{z}\) aligned with \(\hat{n}\), use coordinate xform of previous slide.
- Keen way: Rodrigues’s formula!

\[
x' = x + (\sin \theta) \hat{n} \times x + (1 - \cos \theta) \hat{n} \times (\hat{n} \times x)
\]

- Define “cross product matrix” \(N\):

\[
N = \begin{pmatrix}
0 & -n_3 & n_2 \\
n_3 & 0 & -n_1 \\
-n_2 & n_1 & 0
\end{pmatrix}
\]

so that

\[
Nx = \hat{n} \times x
\]
... using Rodrigues’s formula ...

Substituting the cross product matrix $N$ into Rodrigues’s formula:

$$x' = x + (\sin \theta)Nx + (1 - \cos \theta)N^2x$$

Factoring out $x$:

$$R = I + (\sin \theta)N + (1 - \cos \theta)N^2$$

That’s it! Rodrigues’s formula in matrix form. If you want to you could expand it:

$$
\begin{pmatrix}
    n_1^2 + (1 - n_1^2)c\theta & n_1n_2(1 - c\theta) - n_3s\theta & n_1n_3(1 - c\theta) + n_2s\theta \\
    n_1n_2(1 - c\theta) + n_3s\theta & n_2^2 + (1 - n_2^2)c\theta & n_2n_3(1 - c\theta) - n_1s\theta \\
    n_1n_3(1 - c\theta) - n_2s\theta & n_2n_3(1 - c\theta) + n_1s\theta & n_3^2 + (1 - n_3^2)c\theta
\end{pmatrix}
$$

where $c\theta = \cos \theta$ and $s\theta = \sin \theta$. Ugly.
Converting from $R$ to $\text{rot}(\hat{n}, \theta)$ . . .

- Problem: $\hat{n}$ isn’t defined for $\theta = 0$.
- We will do it indirectly. Convert $R$ to a unit quaternion (next lecture), then to axis-angle.
Quaternion Notes by HyunSoo Park
Recall Euler’s 2D rotation representation

$\exp(i \theta) = \cos \theta + i \sin \theta$

This can be interpreted as a rotation about $Z$ axis with $\theta$. 
A quaternion is defined as:

\[
e = \left[ \begin{array}{c} e_x \\ e_y \\ e_z \end{array} \right]
\]

The exponential form of a quaternion is:

\[
\exp\left( \frac{e}{2} \right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} \left( ie_x + je_y + ke_z \right)
\]

3D rotation via a quaternion is to rotate about e axis with

We will see why \( \theta / 2 \)
Quaternion, inverse of rotation

\[
\begin{align*}
-e &= -\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \\
q^{-1} &= \exp \left( -\frac{e}{2} \right) \\
&= \cos \frac{1}{2} - \sin \frac{1}{2} (ie_x + je_y + ke_z)
\end{align*}
\]
Quaternion’s Topological Space

\[ e_z = 0 \]

\[ e = \begin{bmatrix} e_x \\ e_y \\ 0 \end{bmatrix} \]

Torus: \( S \times S \)
Quaternion's Topological Space

High dimensional torus: $S^2 \times S$

A slice in $X$, $Y$, or $Z$ direction produces 2D torus.

$$e = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$
Exercise

Find \( q \) such that \( q \) describes a rotation of 60 degrees about \( d = [3, 4, 0] \).

\[
q_1 = \cos \frac{60}{2} = \cos \frac{\pi}{3} = \frac{\sqrt{3}}{2}
\]

\[
e = d/\|d\| = i \frac{3}{5} + j \frac{4}{5}
\]

\[
\hat{q} = \sin \frac{60}{2} = \sin \frac{\pi}{3} = \frac{1}{2}
\]

\[
e = \frac{1}{2} \left( i \frac{3}{5} + j \frac{4}{5} \right)
\]

\[
q = \frac{\sqrt{3}}{2} + \frac{1}{2} \left( i \frac{3}{5} + j \frac{4}{5} \right)
\]

\[
\hat{q} = iq_x + jq_y + kq_z
\]

\[
e = d/\|d\| = [3/5, 4/5, 0]^T
\]
Quaternion Algebra

\[ q = \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} e_x + j \sin \frac{\theta}{2} e_y + k \sin \frac{\theta}{2} e_z \]

where \( i^2 = j^2 = k^2 = ijk = -1 \)

Quaternion product

\[ qp = (q_w + iq_x + jq_y + kq_z)(p_w + ip_x + jp_y + kp_z) \]
\[ = (q_wp_w - q_xp_x - q_yp_y - q_zp_z) + i(q_wp_x + q_xp_w + q_yp_z - q_zp_y) \]
\[ + j(q_wp_y - q_xp_z + q_yp_w + q_zp_x) + k(q_wp_z + q_xp_y - q_yp_x + q_zp_w) \]
\[ = (q_wp_w - \hat{q} \cdot \hat{p}) + (q_wp + p_w\hat{q} + \hat{q} \times \hat{p}) \]

where \( \hat{q} = iq_x + jq_y + kq_z \)
Basis element multiplication

From that one axiom, we can derive other products:

\[ ijk = -1 \]
\[ i(ijk) = i(-1) \]
\[ -jk = -i \]
\[ jk = i \]

Writing them all down:

\[ ij = k, \ ji = -k \]
\[ jk = i, \ kj = -i \]
\[ ki = j, \ ik = -j \]

Quaternion products of \( i, j, k \) behave like cross product.
Conjugate, length

Definition (Conjugate)

\[ q^* = q_0 - q_1 i - q_2 j - q_3 k \]

Note that

\[ qq^* = (q_0 + q)(q_0 - q) \]
\[ = q_0^2 + q_0 q - q_0 q - qq \]
\[ = q_0^2 + q \cdot q - q \times q \]
\[ = q_0^2 + q_1^2 + q_2^2 + q_3^2 \]

Definition (Length)

\[ |q| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \]
Quatertion inverse

Every quaternion except 0 has an inverse:

\[ q^{-1} = \frac{q^*}{|q|^2} \]

Without commutativity, quaternions are a division ring, or a non-commutative field, or a skew field.
Just as complex numbers are an extension of the reals, quaternions are an extension of the complex numbers (and of the reals).
If 1D numbers are the reals, and 2D numbers are the complex numbers, then 4D numbers are quaternions, and that’s all there is. (Frobenius) (Octonions are not associative.)
Quatereinion Composition Example

Rotating 90 degrees about $Y$ axis.  
Rotating 90 degrees about $Z$ axis.

$$q_1 = \cos \frac{\theta}{2} + j \sin \frac{\theta}{2}$$

$$q_2 = \cos \frac{\theta}{2} + k \sin \frac{\theta}{2}$$

$$q_{12} = q_1 q_2$$

?
Quaternion Composition Example

\[ q_1 = \frac{\sqrt{2}}{2} + j \frac{\sqrt{2}}{2} \quad q_2 = \frac{\sqrt{2}}{2} + k \frac{\sqrt{2}}{2} \]

\[ q_{12} = q_1 q_2 \]

\[ = \left( \frac{\sqrt{2}}{2} + j \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{2}}{2} + k \frac{\sqrt{2}}{2} \right) \]

\[ = \frac{1}{2} + i \frac{1}{2} + j \frac{1}{2} + k \frac{1}{2} \]

\[ = \frac{1}{2} + \frac{\sqrt{3}}{2} \left( i \frac{1}{\sqrt{3}} + j \frac{1}{\sqrt{3}} + k \frac{1}{\sqrt{3}} \right) \]

\[ \cos \frac{\theta}{2} = \frac{1}{2} \quad \sin \frac{\theta}{2} = \frac{\sqrt{3}}{2} \quad \rightarrow \quad \theta = \frac{2}{3} \]

Rotating about (1,1,1) axis with 120 degrees.
Rotating a Point via Quaternion

Hamiltonian product

Representing a 3D point in quaternion representation with zero angle.

\[ p = 0 + ip_x + jp_y + kp_z \]

Rotation via quaternion:

\[ p' = qpq^{-1} \quad \text{where} \quad q^{-1} = \cos \frac{\theta}{2} - i\sin \frac{\theta}{2} e_x - j\sin \frac{\theta}{2} e_y - k\sin \frac{\theta}{2} e_z \]

\[ p' = qpq^{-1} \]

\[ = \left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \hat{q} \right) p \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \hat{q} \right) \]

\[ = \left( \hat{p} - (\hat{q} \cdot \hat{p}) \hat{q} \right) \cos + (\hat{q} \times \hat{p}) \sin + (\hat{q} \cdot \hat{p}) \hat{q} \]

\[ = \hat{p}_\perp q \cos + \hat{p}_\perp q \sin + \hat{p}_p q \]
Geometric Interpretation

\[ p' = qpq^{-1} \]

\[ = \left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right) p \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right) \]

\[ = \left( p - (\hat{q} \cdot \hat{p}) \hat{q} \right) \cos + (\hat{q} \times \hat{p}) \sin + (\hat{q} \cdot \hat{p}) \hat{q} \]

\[ = \hat{p}_{\perp \hat{q}} \cos + \hat{p}_{\perp \hat{q}} \sin + \hat{p}_{\hat{q}} \]
Example

\[
X \rightarrow Y \quad Y \rightarrow Z \quad Z \rightarrow X
\]

\[
p = i + j
\]

\[
e = \frac{1}{\sqrt{3}}(i + j + k)
\]

\[
q = \frac{1 + i + j + k}{2}
\]

\[
p' = q p q^{-1}
\]

\[
= \left( \frac{1 + i + j + k}{2} \right)(i + j) \left( \frac{1 - i - j - k}{2} \right)
\]

\[
= \frac{1}{4}(i + j - 1 + k - k - 1 + j - i)(1 - i - j - k)
\]

\[
= \frac{1}{4}(2j - 2)(1 - i - j - k)
\]

\[
= \frac{1}{4}(2j + 2k + 2 - 2i - 2 + 2i + 2j + 2k)
\]

\[
= \frac{1}{4}(4j + 4k) = j + k
\]
Quaternion to Rotation Matrix

\[ p' = q p q^{-1} \]

\[ (q_w + i q_x + j q_y + k q_z)(i p_x + j p_y + k p_z)(q_w - i q_x - j q_y - k q_z) \]

\[ = i \left( (1 - 2 q_z q_z - 2 q_y q_y) p_x + (-2 q_z q_w + 2 q_y q_x) p_y + (2 q_y q_w + 2 q_z q_x) p_z \right) \]

\[ + j \left( (2 q_x q_y + 2 q_w q_z) p_x + (1 - 2^* q_z q_z - 2 q_x q_x) p_y + (2 q_z q_y - 2 q_x q_w) p_z \right) \]

\[ + k \left( (2 q_x q_z - 2 q_w q_y) p_x + (2 q_y q_z + 2 q_w q_x) p_y + (1 - 2 q_y q_y - 2 q_x q_x) p_z \right) \]

\[
\begin{bmatrix}
p'_x \\
p'_y \\
p'_z \\
\end{bmatrix} = \begin{bmatrix}
1 - 2 q_z q_z - 2 q_y q_y & -2 q_z q_w + 2 q_y q_x & 2 q_y q_w + 2 q_z q_x \\
2 q_x q_y + 2 q_w q_z & 1 - 2^* q_z q_z - 2 q_x q_x & 2 q_z q_y - 2 q_x q_w \\
2 q_x q_z - 2 q_w q_y & 2 q_y q_z + 2 q_w q_x & 1 - 2 q_y q_y - 2 q_x q_x \\
\end{bmatrix} \begin{bmatrix}
p_x \\
p_y \\
p_z \\
\end{bmatrix}
\]

\[ R = \begin{bmatrix}
\frac{1 - \frac{2}{4}}{4} & \frac{2}{4} & \frac{2}{4} \\
\frac{2}{4} & \frac{1 - \frac{2}{4}}{4} & \frac{2}{4} \\
\frac{2}{4} & \frac{2}{4} & \frac{1 - \frac{2}{4}}{4} \\
\end{bmatrix} \cdot \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix} \]

Permutation matrix
Quaternion to Rotation Matrix

\[ q \times q^* = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \times \]
From rotation matrix to quaternion

Given \( R = (r_{ij}) \), solve expression on previous slide for quaternion elements \( q_i \)

Linear combinations of diagonal elements seem to solve the problem:

\[
q_0^2 = \frac{1}{4}(1 + r_{11} + r_{22} + r_{33})
\]
\[
q_1^2 = \frac{1}{4}(1 + r_{11} - r_{22} - r_{33})
\]
\[
q_2^2 = \frac{1}{4}(1 - r_{11} + r_{22} - r_{33})
\]
\[
q_3^2 = \frac{1}{4}(1 - r_{11} - r_{22} + r_{33})
\]

so take four square roots and you’re done? You have to figure the signs out. There is a better way ...
Look at the off-diagonal elements

\[ q_0 q_1 = \frac{1}{4} (r_{32} - r_{23}) \]
\[ q_0 q_2 = \frac{1}{4} (r_{13} - r_{31}) \]
\[ q_0 q_3 = \frac{1}{4} (r_{21} - r_{12}) \]
\[ q_1 q_2 = \frac{1}{4} (r_{12} + r_{21}) \]
\[ q_1 q_3 = \frac{1}{4} (r_{13} + r_{31}) \]
\[ q_2 q_3 = \frac{1}{4} (r_{23} + r_{32}) \]

Given any one \( q_i \), could solve the above for the other three.
The procedure

1. Use first four equations to find the largest $q_i^2$. Take its square root, with either sign.

2. Use the last six equations (well, three of them anyway) to solve for the other $q_i$.

   - That way, only have to worry about getting one sign right.

   - Actually $q$ and $-q$ represent the same rotation, so no worries about signs.

   - Taking the largest square root avoids division by small numbers.
Axis-angle <-> Rotation Matrix

- Axis-angle -> Rotation of point with Rodrigues formula -> convert to Rotation matrix with same formula

- Rotation Matrix -> Quaternion -> Axis-angle
Rodrigues’s formula for differential rotations

Consider Rodrigues’s formula for a differential rotation $\text{rot}(\hat{n}, d\theta)$.

\[
x' = (I + \sin d\theta N + (1 - \cos d\theta)N^2)x
\]

\[
= (I + d\theta N)x
\]

so

\[
dx = N x \, d\theta
\]

\[
= \hat{n} \times x \, d\theta
\]

It follows easily that differential rotations are vectors: you can scale them and add them up. We adopt the convention of representing angular velocity by the unit vector $\hat{n}$ times the angular velocity.
• https://vimeo.com/49145144

• https://vimeo.com/48948064