Fundamentals of Linear Algebra and Optimization
Active Constraints and Qualified Constraints

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Necessary Condition for Constrained Optimization

Our first main goal of this module is to find a necessary criterion for a function $J\colon \Omega \to \mathbb{R}$ to have a minimum on a subset U defined by *inequality* constraints $\varphi_i(x) \leq 0$, where the functions φ_i are *convex*.

It turns out that the constraints φ_i that matter are those for which $\varphi_i(u) = 0$, namely the constraints that are *tight*, or as we say, *active*.

Definition of an Active Constraint

Definition. Given m functions $\varphi_i \colon \Omega \to \mathbb{R}$ defined on some open subset Ω of some vector space V, let U be the set defined by

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0, \ 1 \leq i \leq m\}.$$

For any $u \in U$, a constraint φ_i is said to be *active* at u if $\varphi_i(u) = 0$, else *inactive* at u if $\varphi_i(u) < 0$.

If a constraint φ_i is active at u, this corresponds to u being on a piece of the *boundary* of U determined by some of the equations $\varphi_i(u) = 0$.

Qualified Constraints

Definition. Let $U \subseteq \Omega \subseteq V$ be given by

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0, \ 1 \leq i \leq m\},$$

where Ω is an open subset of the Euclidean vector space V. If the functions $\varphi_i \colon \Omega \to \mathbb{R}$ are *convex*, we say that the constraints are *qualified* if the following conditions hold:

(a) Either the constraints φ_i are *affine* for *all* $i = 1, \ldots, m$ and $U \neq \emptyset$, or

(b) There is *some* vector $v \in \Omega$ such that the following conditions hold for $i = 1, \ldots, m$:

  (i) $\varphi_i(v) \leq 0$.

  (ii) If φ_i is *not affine*, then $\varphi_i(v) < 0$.

Slater's Conditions

The above qualification conditions are known as *Slater's conditions*.

Condition (b)(i) also implies that U has nonempty relative interior. If Ω is convex, then U is also convex.

Affine Equality Constraints

It is important to observe that a *nonaffine equality constraint* $\varphi_i(u) = 0$ is *never qualified*.

Indeed, $\varphi_i(u) = 0$ is equivalent to $\varphi_i(u) \leq 0$ and $-\varphi_i(u) \leq 0$, so if these constraints are qualified and if φ_i is not affine then there is some nonzero vector $v \in \Omega$ such that both $\varphi_i(v) < 0$ and $-\varphi_i(v) < 0$, which is impossible.

For this reason, *equality constraints are often assumed to be affine*.