Fundamentals of Linear Algebra and Optimization

Convex Sets and Convex Functions

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May 7, 2020

Definition of a Convex Set

Definition. Given any real vector space E, we say that a subset C of E is *convex* if either $C = \emptyset$ or if for every pair of points $u, v \in C$, the line segment connecting u and v is contained in C, i.e.,

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Clearly, a nonempty set C is convex iff $[u, v] \subseteq C$ whenever $u, v \in C$.

Illustration of a Convex Set

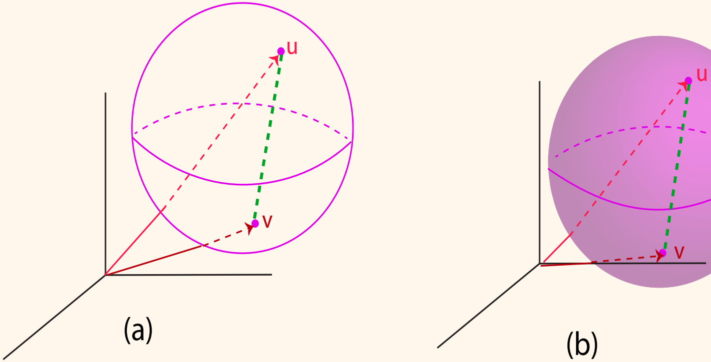

Figure 1: Figure (a) shows that a sphere is not convex in $\mathbb{R}^3$ since the dashed green line does not lie on its surface. Figure (b) shows that a solid ball is convex in $\mathbb{R}^3$.

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the function f is *strictly convex* (on C) if for every pair of distinct points $u, v \in C$ $(u \neq v)$,

$$f((1-\lambda)u + \lambda v) < (1-\lambda)f(u) + \lambda f(v) \quad \text{for all } \lambda \in \mathbb{R} \text{ such that } 0 < \lambda < 1.$$

Epigraph and Convexity

The *epigraph* $\mathbf{epi}(f)$ of a function $f\colon A \to \mathbb{R}$ defined on some subset A of $\mathbb{R}^n$ is the subset of $\mathbb{R}^{n+1}$ defined as

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A function $f\colon C \to \mathbb{R}$ defined on a convex subset C is *concave* (resp. *strictly concave*) if $(-f)$ is convex (resp. strictly convex).

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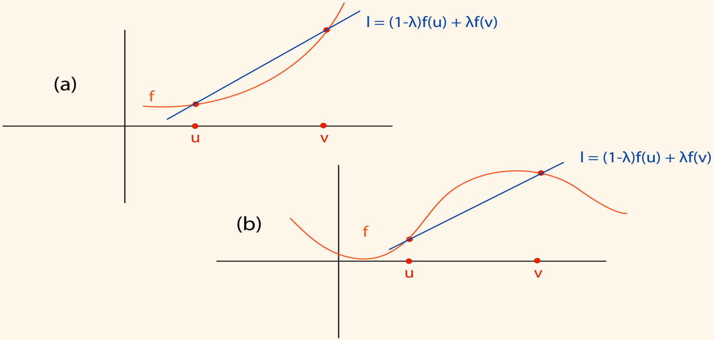

Figure 2: Figures (a) and (b) are the graphs of real valued functions. Figure (a) is the graph of convex function since the blue line lies above the graph of f. Figure (b) shows the graph of a function which is not convex.

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- *Affine subspaces*, that is, sets of the form $u + V$, where V is a subspace of E and $u \in E$, are convex.
- Balls (open or closed) are convex. Given any linear form $\varphi\colon E \to \mathbb{R}$, for any scalar $c \in \mathbb{R}$, the *closed half–spaces*

$$H^{+}_{\varphi,c} = \{u \in E \mid \varphi(u) \geq c\}, \qquad H^{-}_{\varphi,c} = \{u \in E \mid \varphi(u) \leq c\},$$

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- Any intersection of half–spaces is convex.
- More generally, any intersection of convex sets is convex.

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  is convex on $\mathbb{R}^n$.
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- The exponential $x \mapsto e^{cx}$ is strictly convex for any $c \neq 0$ ($c \in \mathbb{R}$).
- The logarithm function is concave on $\mathbb{R}_+ - \{0\}$.
- The *log-determinant function* $\log\det$ is concave on the set of symmetric positive definite matrices. This function plays an important role in convex optimization.

Optimization and Convexity

The following theorem is the key result about the *existence of a local minimum* of a *convex function* with respect to a *convex subset* U.

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(2) Any *strictly convex* function $J \colon U \to \mathbb{R}$ has *at most* one minimum (in U), and if it does, then it is a strict minimum (in U).

(3) Let $J \colon \Omega \to \mathbb{R}$ be any function defined on some open subset Ω of E with $U \subseteq \Omega$ and assume that J is *convex* on U. For any point $u \in U$, if $dJ(u)$ exists, then J has a *minimum* in u with respect to U iff

$$dJ(u)(v-u) \geq 0 \quad \text{for all } v \in U.$$

Optimization and Convexity

(4) If the *convex* subset U in (3) is *open*, then the above condition is equivalent to

$$dJ(u) = 0.$$