DEDUCTION

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1. Natural Deduction Overview

In what follows we present a system of *natural deduction*. For a set of formulas Σ and a formula φ, we will define what it means for $\Sigma \vdash \varphi$. (Note that we are using formulas instead of sentences; the need for this will become clear when we look at deduction rules for quantifiers.)

Recall the definition of formula from before:

(1) If t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
(2) If $t_1, \ldots, t_n$ are terms, and R is an n-ary relation, then $R(t_1, \ldots, t_n)$ is a formula.
(3) If φ is a formula, then $(\neg\varphi)$ is a formula.
(4) If φ, ψ are formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$ are formulas.
(5) If v_i is a variable, and φ a formula, then $\exists v_i(\varphi)$ and $\forall v_i(\varphi)$ are formulas.

We add to this one more thing:

(0) $\top$ and $\bot$ are formulas.[1]

Our goal is to say that $\Sigma \vdash \varphi$ if there is a finite sequence of steps that proves φ under the premises Σ using a number of allowed syntactic manipulations. For each logical symbol, we will have an *introduction rule* and an *elimination rule*. A formal proof basically involves constructing formulas and deconstructing formulas (respectively) using these rules.

There are many ways to define a system of deduction, and even many things that might be called "natural deduction", of which we will only go over one. It ultimately does not matter which system is used.

2. Silly Rules

Premise and Reiteration, while simple, are quite important:

(P) For any φ, $\{\varphi\} \vdash \varphi$.

(R) If $\Sigma \vdash \varphi$, and $\Sigma \subset \Gamma$, then $\Gamma \vdash \varphi$.

Then there is:

($\top$I) For any Γ, $\Gamma \vdash \top$.

($\bot$I) If $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, then $\Gamma \vdash \bot$.

($\bot$E) For any Γ and φ, if $\Gamma \vdash \bot$ then $\Gamma \vdash \varphi$.

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[1] This addition is not strictly necessary. For instance, the notion "$\Sigma \vdash \bot$" (Σ proves false, i.e. Σ is inconsistent) could be expressed instead by "$\Sigma \vdash (\varphi \wedge \neg\varphi)$ for some φ."

We can't deduce anything from $\top$, so we don't have an elimination rule for it. The $\perp$ rules are tied up with the rules for $\neg$ which we will see in a second. Instead of worrying about this sort of relationship, we will just err on the side of having too many rules.

3. Logical Connectives

The basic syntactic rules for $\wedge, \rightarrow, \vee, \neg$ are just as they were for truth-functional logic.

($\wedge$I) If $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$ then $\Gamma \vdash (\varphi \wedge \psi)$.

($\wedge$E) If $\Gamma \vdash (\varphi \wedge \psi)$ then $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.

($\rightarrow$I) If $\Gamma \cup \{\varphi\} \vdash \psi$ then $\Gamma \vdash (\varphi \rightarrow \psi)$.

($\rightarrow$E) If $\Gamma \vdash (\varphi \rightarrow \psi)$ and $\Gamma \vdash \varphi$ then $\Gamma \vdash \psi$.

($\vee$I) If $\Gamma \vdash \varphi$ or if $\Gamma \vdash \psi$ then $\Gamma \vdash (\varphi \vee \psi)$.

($\vee$E) If $\Gamma \vdash (\varphi \vee \psi)$, $\Gamma \cup \{\varphi\} \vdash \theta$ and $\Gamma \cup \{\psi\} \vdash \theta$, then $\Gamma \vdash \theta$.

($\neg$I) If $\Gamma \cup \{\varphi\} \vdash \perp$, then $\Gamma \vdash \neg\varphi$.

($\neg$E) If $\Gamma \vdash \neg\neg\varphi$, then $\Gamma \vdash \varphi$.

Below is an example deduction. By the *scope* we mean the set of current working premises. In other words, $\Gamma \vdash \varphi$ has scope Γ and conclusion φ. Every line of a deduction can be interpreted as a true meta-statement of the form $\Gamma \vdash \varphi$.

Proof that $\{Px \vee Qx, Px \rightarrow Rx, \neg Sx \rightarrow \neg Qx\} \vdash Rx \vee Sx$

Scope	Conclusion	Rule used
1	1. $Px \vee Qx$	P
2	2. $Px \rightarrow Rx$	P
3	3. $\neg Sx \rightarrow \neg Qx$	P
4	4. Px	P
$1, 2, 3, 4$	5. Px	R from 4
$1, 2, 3, 4$	6. $Px \rightarrow Rx$	R from 2
$1, 2, 3, 4$	7. Rx	$\rightarrow E$ from $5, 6$
$1, 2, 3, 4$	8. $Rx \vee Sx$	$\vee I$ from 7
9	9. Qx	P
$1, 2, 3, 9$	10. Qx	R from 9
11	11. $\neg Sx$	P
$1, 2, 3, 9, 11$	12. $\neg Sx$	R from 11
$1, 2, 3, 9, 11$	13. $\neg Sx \rightarrow \neg Qx$	R from 3
$1, 2, 3, 9, 11$	14. $\neg Qx$	$\rightarrow E$ from $12, 13$
$1, 2, 3, 9, 11$	15. Qx	R from 10
$1, 2, 3, 9, 11$	16. $\perp$	$\perp I$ from $14, 15$
$1, 2, 3, 9$	17. $\neg\neg Sx$	$\neg I$ from $12, 16$
$1, 2, 3, 9$	18. Sx	$\neg E$ from 17
$1, 2, 3, 9$	19. $Rx \vee Sx$	$\vee I$ from 18
$1, 2, 3$	20. $Rx \vee Sx$	$\vee E$ from $1, 8, 19$

4. Equality

There are rules for equality, but we won't give any examples.

(=I) If t is a term, then $\Gamma \vdash (t = t)$.

(=E) Let φ' be obtained from φ by replacing zero or more occurences of t with t', and none of the variables of t or t' are bound. If $\Gamma \vdash (t = t')$ and $\Gamma \vdash \varphi$, then $\Gamma \vdash \varphi'$.

5. Quantifiers

($\forall$I) If $\Gamma \vdash \varphi$ and the variable v does not occur free in Γ, then $\Gamma \vdash (\forall v)\varphi$.

($\forall$E) Let φ' be obtained from φ by replacing all free instances of v in φ with a term t, such that all variables of t are free in φ'. If $\Gamma \vdash (\forall v)\varphi$ then $\Gamma \vdash \varphi'$.

($\exists$I) Let φ' be obtained from φ by replacing all free instances of v in φ with a term t, such that all variables of t are free in φ'. If $\Gamma \vdash \varphi'$, then $\Gamma \vdash (\exists v)\varphi$.

($\exists$E) If $\Gamma \vdash (\exists v)\varphi$ and $\Gamma \cup \{\varphi\} \vdash \psi$, and the variable v does not occur free in Γ or in ψ, then $\Gamma \vdash \psi$.

Here is first a bad deduction and then two good deductions using these rules.

Proof that $\{(\exists x)Px\} \vdash (\forall x)Px$

Scope	Conclusion	Rule used
1	1. $(\exists x)Px$	P
2	2. Px	P
$1, 2$	3. Px	R from 2
$1, 2$	4. $(\forall x)Px$	$\forall I$ from 3 (BAD)
1	5. $(\forall x)Px$	$\exists E$ from $1, 4$

Line 4 is bad because the variable x occurs free in the current set of premises (1,2). All other lines are okay.

Proof that $\{(\exists x)(Px \wedge Qx), (\forall x)\neg Px\} \vdash \bot$

Scope	Conclusion	Rule used
1	1. $(\exists x)(Px \wedge Qx)$	P
2	2. $(\forall x)\neg Px$	P
$1, 2$	3. $(\exists x)(Px \wedge Qx)$	R from 1
$1, 2$	4. $(\forall x)\neg Px$	R from 2
$1, 2$	5. $\neg Px$	$\forall E$ from 4
6	6. $Px \wedge Qx$	P
$1, 2, 6$	7. $Px \wedge Qx$	R from 6
$1, 2, 6$	8. Px	$\wedge E$ from 7
$1, 2, 6$	9. $\neg Px$	R from 5
$1, 2, 6$	10. $\bot$	$\bot I$ from $8, 9$
$1, 2$	11. $\bot$	$\exists E$ from $3, 10$

Proof that $\{(\forall x)(\forall y)Pxy, (\exists x)(Pxx \to Qxx)\} \vdash (\exists x)Qxx$

Scope	Conclusion	Rule used
1	1. $(\forall x)(\forall y)Pxy$	P
2	2. $(\exists x)(Pxx \to Qxx)$	P
$1,2$	3. $(\forall x)(\forall y)Pxy$	R from 1
$1,2$	4. $(\exists x)(Pxx \to Qxx)$	R from 2
$1,2$	5. $(\forall y)Pxy$	$\forall E$ from 3
$1,2$	6. Pxx	$\forall E$ from 5
7	7. $Pxx \to Qxx$	P
$1,2,7$	8. $Pxx \to Qxx$	R from 7
$1,2,7$	9. Pxx	R from 6
$1,2,7$	10. Qxx	$\to E$ from $8,9$
$1,2,7$	11. $(\exists x)Qxx$	$\exists I$ from 10
$1,2$	12. $(\exists x)Qxx$	$\exists E$ from $4,11$

In all of our examples, when we use the Reiteration rule R, we have only added finitely many premises. Moreover, so far we have only added premises which were already stated via the premise rule. We could, however, add infinitely many premises, and any premises we want, though we would have to notate this a bit differently:

Proof that $\Gamma \cup \{Px\} \vdash Px$

Scope	Conclusion	Rule used
1	1. Px	P
$\Gamma, 1$	2. Px	R from 1

Here Γ could be any set. Usually, we use shorthand $1, 2, 3, \dots$ to refer to premises by their line number instead of writing them out explicitly.

6. Consistency and Satisfiability

For most of this lecture, Γ was a set of formulas. While we could define $\Gamma \vDash \varphi$ as well for Γ any set of formulas, we only really care about theories (sets of sentences), so we now restrict to that particular case.

A theory $\mathcal{T}$ is *inconsistent* if $\mathcal{T} \vdash \bot$. Otherwise, it is *consistent*.

A theory $\mathcal{T}$ is *unsatisfiable* if it has no models. If it has at least one, it is *satisfiable*.

Soundness says that if $\mathcal{T} \vdash \varphi$ then $\mathcal{T} \vDash \varphi$.

Completeness says that if $\mathcal{T} \vDash \varphi$ then $\mathcal{T} \vdash \varphi$.

Soundness is easy to prove, but completeness is hard, due to Godel.

An elementary consequence of soundness and completeness is that a theory is satisfiable if and only if it is consistent. Another consequence is the compactness theorem, which is forthcoming.