FILTERS AND ULTRAFILTERS

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1. Filters

Given a set X, a filter is a way of calling certain subsets of X equivalent. We will say two subsets of X are equivalent if the set of $x \in X$ where the subsets agree is in the filter. For now, let's just state the definition.

Definition 1. Let X be a set. A **filter** is a set $\mathcal{F}$ with the following properties:

(1) (nonempty) $X \in \mathcal{F}$
(2) (intersections) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
(3) (upper set) If $A \in \mathcal{F}$ and $B \supset A$, then $B \in \mathcal{F}$.

Given a filter $\mathcal{F}$, we say that

Definition 2. $\mathcal{F}$ is **free** if $\cap \mathcal{F} = \varnothing$.

Definition 3. $\mathcal{F}$ is **principal** if there exists a set S, $S \neq \varnothing$ and $S \subseteq X$, such that $\mathcal{F} = \{A : S \subseteq A\}$. $\mathcal{F}$ is **nonprincipal** otherwise.

Definition 4. $\mathcal{F}$ is **proper** if $\varnothing \notin \mathcal{F}$.[1]

The basic examples of a filter is a principal filter, formed by taking a set S and considering all sets containing it. Principal filters, however, are in some sense trivial and will be uninteresting to us. We are really interested in free filters (and in particular free ultrafilters); these will allow us to explicitly construct structures out of smaller structures in a very interesting way.

The canonical example of a free filter is the set of cofinite subsets of X. This is called the **Frechet filter** on X.

Here are some important propositions:

Proposition 1. *Every free filter is nonprincipal.*

Proof. Let X be a set and $\mathcal{F}$ a filter on X. Suppose by way of contrapositive that $\mathcal{F}$ is principal. Then $\mathcal{F} = \{A : A \supseteq S\}$, and in particular $S \in \mathcal{F}$, so that clearly $\cap \mathcal{F} = S$. But S is nonempty by the definition of a principal filter, so $\mathcal{F}$ is not free. □

The converse doesn't hold; nonprincipal filters exist which are not free. There are some examples of this in the next section.

Proposition 2. *Let $\mathcal{F}$ be a filter on X. TFAE:*

1. $\mathcal{F}$ is free.

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[1]In the Pete Clark notes, and some other sources, filters are required to be proper in the definition. There is only one filter which isn't proper; the set of all subsets of X. Wikipedia allows this trivial filter, so I have allowed it here.

2. $\mathcal{F}$ contains the Frechet filter.

Proof. ($1 \implies 2$) Let $\mathcal{F}$ be free. For each $x \in X$, since the intersection of all sets in $\mathcal{F}$ is empty, there must be a set $A_x \in \mathcal{F}$ such that $x \notin A_x$. The set $X \setminus \{x\}$ in turn contains A_x, so $X \setminus \{x\} \in \mathcal{F}$. Any cofinite set is the intersection of sets of the form $X \setminus \{x\}$, so all cofinite sets must therefore be in $\mathcal{F}$.

($2 \implies 1$) Let F be the Frechet filter. Note that $\mathcal{F} \supset F$ implies $\cap\mathcal{F} \subset \cap F = \varnothing$, so $\cap\mathcal{F} = \varnothing$. □

Another thing about filters is they are really only interesting to us in the case that the underlying set is infinite.

Proposition 3. *Let X be a finite set and $\mathcal{F}$ a proper filter on X. Then $\mathcal{F}$ is principal.*

Proof. Let $S = \cap\mathcal{F}$. Since X is finite, $\mathcal{P}(X)$ is finite, so this is a finite intersection; hence $S \in \mathcal{F}$. Then $\mathcal{F} = \{A : S \subseteq A\}$. Since $\mathcal{F}$ is propert, S is nonempty. □

2. Examples

Here's a big list of examples. Make sure you understand the classification in each case.

$\boldsymbol{X}$		**Subset of $\mathcal{P}(X)$**	**Filter?**	**Principal?**	**Free?**
$\{0,1\}$	1.	$\{0\},\{0,1\}$	Y	Y	N
	2.	$\{0,1\}$	Y	Y	N
	3.	$\{0,1\},\{0\},\{1\}$	N		
$\mathbb{N}$	4.	$\{S : \lvert S\rvert < \infty\}$	N		
	5.	$\{S : \lvert\mathbb{N} \setminus S\rvert < \infty\}$	Y	N	Y
	6.	$\{S : \lvert\mathbb{N} \setminus S\rvert < \infty, 0 \in S\}$	Y	N	N
	7.	$\{S : \lvert\mathbb{N} \setminus S\rvert < \infty, 0 \notin S\}$	N		
	8.	$\{S : 31 \in S\}$	Y	Y	N
	9.	$\{S : 2, 4, 6, 8, \ldots \in S\}$	Y	Y	N
$\mathbb{R}$	10.	$\{S : 0 \in \text{Int}S\}$	Y	N	N
	11.	$\{S : 0 \in S\}$	Y	Y	N
	12.	$\{S : \text{Int}S \neq \varnothing\}$	N		
	13.	$\{S : S \text{ open}, S \neq \varnothing\}$	N		
	14.	$\{S : \mathbb{R} \setminus S \text{ is bounded}\}$	N		
	15.	$\{S : \mathbb{R} \setminus S \text{ has measure } 0\}$	Y	N	Y
	16.	$\{S : \mathbb{R} \setminus S \text{ has finite outer measure}\}$	Y	N	Y

Example 5 is the Frechet filter on $\mathbb{N}$. Examples 6 and 10 illustrate the character of nonprincipal filters which are not free: they would be free, except that a certain element (or multiple elements) are required to be in every set of the filter.

3. Ultrafilters

Definition 5. Let X be a set. An **ultrafilter** on X is a filter $\mathcal{U}$ which additionally satisfies:

(4) (proper) $\varnothing \notin \mathcal{U}$
(5) (maximal) For all $S \subseteq X$, either $S \in \mathcal{U}$ or $X \setminus S \in \mathcal{U}$.

An equivalent definition is that for all $S \subseteq X$, either $S \in \mathcal{U}$ or $X \setminus S \in \mathcal{U}$, but not both. The "but not both" implies that $\mathcal{U}$ doesn't contain the empty set, and vice versa.

Of the examples in the previous section, $1, 8$, and 11 are ultrafilters. But all of these are *principal* ultrafilters, and seem quite trivial. In fact, the following holds:

Proposition 4. *Let X be a set and $\mathcal{U}$ be an ultrafilter on X. TFAE:*

1. *$\mathcal{U}$ is not free.*
2. *$\mathcal{U}$ is principal.*
3. *$\mathcal{U} = \{S : x \in S\}$ for some $x \in X$.*

Proof. Homework. □

So, principal ultrafilters are just those generated by a single element. Why didn't we give any examples of nonprincipal ultrafilters? Because their existence turns out to be equivalent to a weaker version of the axiom of choice known as the **ultrafilter lemma**. As a result, they can't be constructed explicitly.

Theorem 1 (Ultrafilter Lemma)**.** *Let X be a set and $\mathcal{F}$ be a filter on X. If $\mathcal{F}$ is proper, then there is an ultrafilter on $\mathcal{U}$ containing X.*

Proof. Order proper filters by inclusion. Show that every chain of filters has an upper bound. Apply the lemma of Zorn and show the resulting filter is an Ultrafilter. □

Corollary 1. *Let X be an infinite set. Then there exists a nonprincipal ultrafilter on X.*

Proof. Let F be the Frechet filter on X. Since X is infinite, $\varnothing$ is not cofinite, so F is proper. By the ultrafilter lemma, there is thus an ultrafilter $\mathcal{U} \supset F$.

By proposition 2, since $\mathcal{U}$ contains the Frechet filter, it is free. By proposition 4 (or by the weaker proposition for filters), since $\mathcal{U}$ is free, it is nonprincipal. □

Note that the Ultrafilter lemma is easy in the case that the filter to be completed is not free; just take any element in the intersection of the filter, and take the principal ultrafilter generated by that element. Zorn's lemma is required only to construct nonprincipal ultrafilters.

4. Bulding structures from structures

In the past, we have used compactness and the Lowenheim-Skolem theorems to construct models for various theories. But in every case, the resulting models were not explicit. What if we want to build explicit models?

Given an index set I and $\mathcal{L}$-structures $\{X_i : i \in I\}$, one obvious thing to do is take the cartesian product, $X := \prod_{i \in I} X_i$. This is an $\mathcal{L}$-structure under the interpretation where function symbols are applied to each individual coordinate, and relation symbols hold exactly when they hold in every individual coordinate.

The problem is that, supposing $X_i \vDash \mathcal{T}$ for all i, for some theory $\mathcal{T}$, we don't necessarily have $X \vDash \mathcal{T}$, which we would like to be true. For example, this does turn out to be true in the theory of groups and the theory of rings (as the cartesian product of groups or rings is itself a group or ring), but this doesn't work when we have the theory of integral domains, or fields, or totally ordered sets.

The solution is to first form the cartesian product, and then to "mod out" by an ultrafilter. If the set I is infinite and the ultrafilter is nonprincipal, we will get (by black magic) a model which satisfies exactly the same formulas as X_i, if all X_i were the same. If the X_i are different, it is slightly more complicated, but we can still say exactly what formulas are satisfied. This result is known as Los's theorem, and is otherwise known as the fundamental theorem of ultrafilters. We will get to it next time.